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ABSTRACT

This document is the report of a curriculum writing session, and covers three activities: (1) a textbook for grade seven, written using the outlines produced the previous year (SE 012 735), was reviewed and criticized; (2) the outlines for grades eight and nine, also produced the previous year, were completed in greater detail; and (3) considerable attention was given to possible sequences of courses for grades 10-12. Broad outlines for several courses at this level were written, including a deductive block; a year of vector geometry, linear algebra, and elementary functions for grade 10; a course in analytic geometry and algebra; and a course in elementary functions and calculus. Several papers treating specific topics are included in this report. (MM)

ED 059876

SCHOOL MATHEMATICS
STUDY GROUP

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TENTATIVE OUTLINES
# of a
SECONDARY SCHOOL
MATHEMATICS CURRICULUM
VOLUME II

SMSG Working Paper

SE 012 734

September, 1967

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Preface

In the summer of 1966, a committee of mathematicians and mathematics teachers met for four weeks to formulate preliminary recommendations for the curricular experimentation which the SMSG plans to carry on during the next few years.

During the academic year, 1966-67, a team of four writers used the Grade 7 outline produced by the summer outlining committee as a starting point in writing a text for that level. The text that evolved showed that some of the outlines were adhered to without change, some of the outlines were revised, and some replaced by newly constructed outlines.

A team of mathematicians and mathematics teachers met in a three-week session which began July 3, 1967, to continue the work started the summer before. The majority of the members in this 1967 committee had also participated in the outlining session in 1966. In carrying out the functions of formulating and revising further recommendations for curricular experimentation, a review was made of the Grade 7 text that had been written. Reports and reactions to the various chapters in the text were discussed. From the discussions, comments, and constructive criticisms of the Grade 7 text, came many suggestions: (1) for teachers trying out the text 1967-68, (2) for revision of the text, (3) for moving some of the material to later years, and (4) for revision of the outlines for Grades 8 and 9.

This report contains suggestions where material moved from Grade 7 should go and contains detailed outlines for the chapters in Grades 8 and 9. A document of helpful hints for teachers trying out the text and a document for the revision team was produced, but these documents are reproduced separately. One section of this report shows the 1966 tentative chapter headings and the 1967 revised chapter headings side by side so that quick comparisons can be made.

The completion of the detailed outlines for Grades 8 and 9 was only part of the task. At the same time, much work and thought were being given to the possible sequences of courses for Grades 10-12 for the various types of college capable students. One of the documents spells out many possible sequences.

There was considerable feeling in the group that careful attention must be given to pedagogy in the writing. Another center of emphasis was in using "real life situations that require mathematics" as motivating factors. Some documents illustrate these points.

The remainder of this report gives broad tentative outlines of what might be included in the Grades 10-12 block. First come several suggestions for a one-semester deductive block. This is followed by three different suggestions for two or three semesters of vector geometry, linear algebra, and elementary functions. Although there was not complete agreement that the senior year should be elementary functions and calculus, that is the outline that was produced.

Three appendices complete this report. Appendix A is a collection of geometry problems that hopefully will be sprinkled through the Grades 10-12 sequence. Appendix B keeps the idea of modeling before the outliners and writers. Appendix C shows how a chapter on systems of linear equations and inequalities might be introduced using a linear programming problem.

List of Participants

Mrs. Pamela Ames - University of Chicago
E. G. Begle - Director, School Mathematics Study Group
Max S. Bell - University of Chicago
Jean M. Calloway - Kalamazoo College
Clyde L. Corcoran - California High School, Whittier
Richard A. Dean - California Institute of Technology
W. Eugene Ferguson - Newton High School, Newtonville, Massachusetts
Herbert J. Greenberg - University of Denver
Mario L. Juncosa - The Rand Corporation
William G. Lister - University of New York - Stony Brook
H. Stewart Moredock - Sacramento State College
Lowell J. Paige - University of California at Los Angeles
Donald E. Richmond - Williams College
William Wernick - City College of New York
Miss Martha Zelinka - Weston High School, Weston, Massachusetts

William G. Chinn - School Mathematics Study Group, attended the plenary sessions.

Miscellaneous Items of Information

1. A separate document has been prepared for the use of teachers who will be trying out the Grade 7 materials, and other people directly concerned with this tryout.

2. A separate document has been prepared for the use of the team that will be revising the Grade 7 materials, and other people directly concerned with the revision.

3. The Grade 7 writing team produced a separate document called "Suggestions on Where to go with Flow Charting in Grades 8-9" which is available for the use of the writing teams.

Contents of Grade 7

Chapter 1 - The Structure of Space

1-1. Introduction
1-2. Points
1-3. Space
1-4. Lines
1-5. Planes
1-6. Intersections
1-7. Intersections of Lines and Planes
1-8. Segments
1-9. Separations
1-10. Angles
1-11. Triangles
1-12. Curves, Simple Closed Curve
1-13. Convexity
1-14. Orientation on a Line
1-15. Orientation in the Plane
1-16. Orientation in Space
1-17. Summary

Chapter 2 - Graphing

2-1. Locating Points in a Plane
2-2. Coordinates on a Line
2-3. Coordinates in the Plane
2-4. Graphs in the Plane

Chapter 3 - Functions

3-1. Introduction
3-2. Some Examples of Functions
3-3. Methods of Representing Functions
3-4. Interpretations from Graphs
3-5. The Identity Function
3-6. Step Functions

Chapter 4 - Informal Algorithms and Flow Charts

4-1. Informal Algorithms and Flow Charts
4-2. Algorithms, Flow Charts and Computers
4-3. Assignment and Variables
4-4. Using a Variable as a Counter
4-5. Decision and Branching
4-6. Flow Charting and Division Algorithm
4-7. Making the Division Algorithm Practical

Chapter 5 - Rational Numbers

5-1. Negative Numbers
5-2. Opposite and Signum Functions
5-3. Absolute Value Functions
5-4. Maximum
5-5. Addition of Rational Numbers
5-6. Flow Chart for Addition
5-7. Functions Using Addition
5-8. Products with One Negative Factor
5-9. Multiplying Negative Numbers
5-10. More Products of Negative Numbers
5-11. Multiplicative Inverse
5-12. Subtraction
5-13. Division
5-14. Computations Using Rational Numbers

Chapter 6 - Structure

6-1. The Rational Number System
6-2. Fields
6-3. The Minus Sign
6-4. Order
6-5. Betweenness

Chapter 7 - Equations and Inequalities

7-1. Introduction
7-2. Equations as Models of Real Life Problems
7-3. A Systematic Method of Solution

7-4. Streamlining Our Method of Solving
7-5. Simplifying
7-6. Inequalities and Lemonade
7-7. Graphical Solution of Equations and Inequalities

Chapter 8 - Congruence

8-1. A Road Building Problem
8-2. Congruent Segments and Congruent Angles
8-3. Copying Triangles
8-4. Congruent Triangles and Correspondence
8-5. Some Applications of Congruence
8-6. Congruent Figures and Motions
8-7. Translations in the Plane
8-8. Rotation
8-9. Reflection
8-10. Congruence of a Figure with Itself
8-11. Writing a Proof
8-12. Addition and Subtraction Properties for Segments
8-13. Addition and Subtraction Properties for Angles
8-14. Bisectors and Perpendiculars
8-15. Construction of Rhombuses
8-16. A Useful Property of the Rhombus
8-17. A Shortest Path Problem
8-18. Medians of a Triangle
8-19. Angle Bisectors of the Angles of a Triangle
8-20. Altitudes of a Triangle
8-21. Perpendicular Bisectors of the Sides of a Triangle

Chapter 9 - Number Theory

9-1. Pails of Water
9-2. Common Factors
9-3. Prime Factorization
9-4. The Euclidean Algorithm
9-5. Pails of Water Again
9-6. A Representation of the Greatest Common Divisor
9-7. A Fundamental Theorem
9-8. Summary

Chapter 10 - Measure

10-1. Measurement
10-2. Standard Units of Length
10-3. Approximation
10-4. Circle, Radius, Diameter
10-5. The Number, π
10-6. Applications Involving π
10-7. Angle Measurement
10-8. Sum of Measures of Angles of Triangle
10-9. Applications
10-10. Measure of Central Angle of a Circle
10-11. Equivalence of Regions
10-12. Regions Equivalent to Regions Bounded by Regular Polygons
10-13. Rectangular Region Equivalent to a Given Region
10-14. Equivalence of Regions in Space
10-15. Comparison of Measures of Set and Subset

Chapter 11 - Probability

11-1. Introduction
11-2. Uncertainty
11-3. Fair and Unfair Games
11-4. Finding Probabilities
11-5. Counting Outcomes; Tree Diagrams
11-6. Estimating Probabilities
11-7. Probability of Union
11-8. Mutually Exclusive Events
11-9. Probability of Intersection

Chapter 12 - Parallelism

12-1. Parallel Lines in the Plane
12-2. Rectangles
12-3. Transversals
12-4. Parallels and the Circumference of the Earth
12-5. Triangles
12-6. Parallelograms
12-7. More About Parallelograms

12-8. Families and Networks of Parallel Lines
12-9. Dividing Segments into Congruent Parts
12-10. Triangles and Families of Parallel Lines
12-11. Networks and Coordinates
12-12. Parallels in Space

Chapter 13 - Similarity

13-1. Scale Drawings
13-2. Similar Triangles
13-3. Multiplying Geometrically
13-4. How a Photo Enlarger Works
13-5. Similarities in Right Triangles
13-6. Slope
13-7. Lines
13-8. Parallel and Perpendicular Lines
13-9. The Meaning of Percent
13-10. Using Percent in Solving Problems
13-11. Summary

Chapter 14 - Real Numbers

14-1. Distance
14-2. Pythagorean Property
14-3. Proof of Pythagorean Property
14-4. Back to Distance
14-5. Real Numbers
14-6. Square Roots and the Pythagorean Property
14-7. Operations with Real Numbers
14-8. Decimal Representations of Real Numbers
14-9. Decimal Representation for Rational Numbers
14-10. Periodic Decimals
14-11. Summary

1966 and Proposed 1967 Sequence of Chapter Headings for Grades 7, 8, and 9

Grade 7 (1966)

1. The Structure of Space -- Nonmetrical Properties
2. Graphs, Functions, Variables
3. The Positive Rationals
3. The Set of Rationals (Alternate Version)

$3\frac{1}{2}$. The Set of Rationals, Solution of Mathematical Sentences (Alternate Version (continued))

4. Congruence (Replication of Figures)
5. Measure
6. Ratio and Similarity
7. Combinatorics and Probability
8. Rational Numbers, Graphs of Functions
8. Graphs of Linear Functions: Variation (Alternate Version) (To accompany "Alternate Versions" of Chapters 3, $3\frac{1}{2}$)
9. Solutions of Systems of Equations and Inequalities
10. Decimals, Square Roots, Real Number Line
11. Parallelism

Grade 7 (1967)

1. The Structure of Space (Suggested, 1-12 and 1-13 be moved to Grade 8, Ch. 9, and 1-14, 1-15, 1-16 be moved to Grade 8, Ch. 4)
2. Graphing
3. Functions
4. Informal Algorithms and Flow Charts

$4\frac{1}{2}$. Applications of Mathematics and Mathematical Models

5. Rational Numbers
6. Structure (Suggested, 6-1, 6-2 be moved to Grade 9, Ch. 12; a lighter treatment should be given here.)
7. Equations and Inequalities
8. Congruence (Suggested, 8-7, 8-8, 8-12, 8-13, 8-18, 8-19, 8-20, 8-21 be moved to Grade 8, Ch. 4)
9. Number Theory
10. Measure
11. Probability
12. Parallelism (Suggested, 12-4, 12-7 through 12-11 be moved to Grade 8, Ch. 9)

Grade 8 (1966)

1. Perpendicularity
2. Coordinate Systems - Distance
3. Displacements (Suggested, move to Grade 9, Ch. 6)
4. Problem Analysis (Strategies)
5. Number Theory (Moved to Grade 7, Ch. 9)
6. The Real Numbers Revisited - Radicals
7. Truth Sets of Mathematical Sentences
8. Quadratic Polynomials as Functions
9. Probability (No outline in 1966)
10. Parallels and Perpendiculars
11. Properties and Mensuration of Geometric Figures
12. Spatial Perception and Locus
13. Systems of Equations in Two Variables

Grade 7 (1967)

13. Similarity (Suggested, 13-3, 13-4 be moved to Grade 8, Ch. 4)
14. Real Numbers (The material in 6-1 and 6-2 on the Rational Number System and Fields should be touched lightly here. To wait until Grade 9, Chapter 12 to talk about extending the number systems is too late.)

Grade 8 (1967)

1. Perpendicularity
2. Coordinate Systems - Distance
3. Problem Analysis
4. Congruence and Similarity (Some of material to come from Grade 7, Chapters 1, 8, 13 as indicated under Grade 7 (1967))
5. The Real Numbers Revisited - Radicals (A little on structure of the system from Grade 7, Ch. 6, 6-1 and 6-2 might come in here.)
6. Truth Sets of Mathematical Sentences
7. Quadratic Polynomials as Functions
8. Probability
9. Parallels and Perpendiculars (Include some of material in Grade 7 (1967), Chapters 1, 12.)
10. Properties and Mensuration of Geometric Figures

Grade 9 (1966)

1. Exponents, Logarithms, Slide Rule
2. Transformations
3. Systems of Sentences
   (No outline in 1966)
4. Systems of Sentences
   (No outline in 1966)
5. Measure Theory
6. Statistics
   (No outline in 1966)
7. Deductive Reasoning
8. Vectors
   (No outline in 1966)
9. Circular Functions
10. Tangency
11. Measure
12. Complex Numbers
    (No outline in 1966)

Grade 8 (1967)

11. Spatial Perception and Locus
12. Systems of Sentences in Two Variables

Grade 9 (1967)

1. Exponents, Logarithms, Slide Rule (1966)
2. Deductive Reasoning - Logic
3. Systems of Sentences
4. Measure Functions and Their Properties (1966)
5. Statistics
6. Displacements - Vectors
   (Displacement Chapter from 1966 Grade 8, Ch. 3) This needs some extension depending on the requirements for the Chapter on Vectors and Analytic Geometry or whatever the next chapter involving vectors will be called)
7. Transformations (1966)
8. Circular Functions (1966)
9. Tangency (1966)
10. Measure (1966)
11. Complex Numbers
    (This Chapter should include some of the material in Grade 7, Ch. 6, Sect. 6-1 and 6-2. A lighter treatment of 6-1 and 6-2 should appear in Grade 7 and Grade 8 before this Chapter on complex numbers. The idea of

Grade 9 (1967)

extending the number system should not be new to the student at this point.)

Location of Topics in SMSG Intermediate Mathematics in the New Outlines

	Chapter and Topics	Location in New Outline (1967)
	Chapter 1 - Number Systems	
	1-1 to 1-7 The Rationals	Gr. 7, Ch. 5
	1-8 Decimal Representation of Rationals	Gr. 7, Ch. 5, 14
	1-9 Infinite Decimals and Real Numbers	Gr. 7, Ch. 14
	1-10 The Equation $x^n = a$ (Exponents, Radicals)	Gr. 8, Ch. 5; Gr. 9, Ch. 1
(*)	1-11 Polynomials and Their Factors	Gr. 8, Ch. 7 (Quad. only)
*	1-12 Rational Expressions	
	Chapter 2 - Introduction to Coordinate Geometry	
	2-1 The Coordinate System (Review)	Gr. 7, Ch. 2; Gr. 8, Ch. 2
	2-2 Distance Between Two Points	Gr. 8, Ch. 2
	2-3 Slope of a Line	Gr. 8, Ch. 1
(*)	2-4 Sketching Graphs of Equations and Inequalities	(Some in Gr. 8, Ch. 6 - Most omitted)
*	2-5 Analytic Proofs of Geometric Theorems	
*	2-6 Sets Satisfying Geometric Conditions	
	Chapter 3 - The Function Concept and the Linear Function	Gr. 7, Ch. 2, 5, 11 (Treatment not comparable, review probably needed)
	Chapter 4 - Quadratic Functions and Equations	
	4-1 to 4-11	Gr. 8, Ch. 7
*	4-12 Some Properties of Roots of Quadratic Equations	
	4-13 Equations Transformable to Quadratic Equations	Gr. 8, Ch. 7

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* Not yet covered in Outlines for 7-9.

Chapter and Topics	Location in New Outline (1967)
Chapter 4 - Continued	
* 4-14 Quadratic Inequalities	
* 4-15 Applications	
Chapter 5 - Complex Number Systems	
5-1 to 5-8	Gr. 9, Ch. 11
* 5-9 Polynomial Equations	
* 5-10 Miscellaneous Exercises	
5-11 Construction of the Complex Number System	
Chapter 6 - Equations of First and Second Degree in Two Variables	
* 6-1 The Straight Line	(Not in this form)
* 6-2 The General Linear Equation $Ax + By + C = 0$	(Not in this form)
* 6-3 The Parabola	
* 6-4 The General Definition of the Conic	
* 6-5 The Circle and the Ellipse	
* 6-6 The Hyperbola	
Chapter 7 - Systems of Two Equations in Two Variables	
(*) 7-1 Solution Sets of Systems of Equations and Inequalities	Partly in Gr. 8, Ch. 12
7-2 Equivalent Equations and Equivalent Systems of Equations	Gr. 9, Ch. 3
7-3 Systems of Linear Equations	Gr. 9, Ch. 3
* 7-4 Systems of One Linear and One Quadratic Equation	
* 7-5 Other systems	

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* Not yet covered in Outlines for 7-9.

Chapter and Topics			Location in New Outline (1967)
*Chapter 8 -		Systems of First Degree Equations in Three Variables	
*Chapter 9 -		Logarithms and Exponents	Gr. 9, Ch. 1 (Not at all comparable coverage)
Chapter 10 -		Introduction to Trigonometry	
*	10-1	Arcs and Paths	
*	10-2	Signed Angles	
	10-3	Radian Measure	Gr. 9, Ch. 8
*	10-4	Other Angle Measures	
	10-5	Definition of Trigonometric Functions	Gr. 9, Ch. 8
*	10-6	Some Basic Properties of Sine and Cosine	
	10-7	Trigonometric Functions of Special Angles	Gr. 9, Ch. 8
	10-8	Tables of Trigonometric Functions	
	10-9	Graphs of Trigonometric Functions	Gr. 9, Ch. 8
*	10-10	Law of Cosines	
*	10-11	Law of Sines	
*	10-12	Addition Formulas	
*	10-13	Identities and Equations	
Chapter 11 -		The System of Vectors	Gr. 9, Ch. 6
	11-1	Directed Line Segments	Gr. 9, Ch. 6
*	11-2	Applications to Geometry	
	11-3	Vectors and Scalars; Components	Gr. 9, Ch. 6
*	11-4	Inner Product	
*	11-5	Applications to Physics	
*	11-6	Vectors as a Formal Mathematical System	

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* Not yet covered in Outlines for 7-9.

Grade 8 - Listing of Chapters

Grade 8 (1966)

1. Perpendicularity
2. Coordinate Systems - Distance
3. Displacements
4. Problem Analysis (Strategies)
5. Number Theory
6. The Real Numbers Revisited - Radicals
7. Truth Sets of Mathematical Sentences
8. Quadratic Polynomials as Functions
9. Probability
10. Parallels and Perpendiculars
11. Properties and Mensuration of Geometric Figures
12. Spatial Perception and Locus
13. Systems of Equations in Two Variables

Grade 8 (1967)

1. Perpendicularity
2. Coordinate Systems - Distance
3. Problem Analysis
4. Congruence and Similarity
5. The Real Numbers Revisited - Radicals
6. Truth Sets of Mathematical Sentences
7. Quadratic Polynomials as Functions
8. Probability
9. Parallels and Perpendiculars
10. Properties and Mensuration of Geometric Figures
11. Spatial Perception and Locus
12. Systems of Sentences in Two Variables

Chapter and Topics	Location in New Outline (1967)
Chapter 12 - Polar Form of Complex Numbers	Gr. 9, Ch. 11 (?)
*Chapter 13 - Sequences and Series	Gr. 12, Ch. 1 (?)
*Chapter 14 - Permutations and Combinations and the Binomial Theorem	(Prob. Stat. Semester)
*Chapter 15 - Algebraic Structures	

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* Not yet covered in Outlines for 7-9.

Notes for Grade 8 Writing Team

1. It has been suggested that Grade 8 (1966 Outline), Chapter 3, Displacements, be moved to Grade 9, Chapter 6. This seems to be a good move, but the article "Vectors in the 7-9 Outline" which follows these notes, pp. 17, raises some questions that must be resolved by the writing team. If the total amount of material to be taught in Grade 8 still allows some time for the Displacement Chapter then the group felt that at least an introduction should be given in Grade 8. In that case, Grade 9, Chapter 6, Displacements - Vectors, could be extended somewhat to prepare for the treatment of vectors and analytic geometry in the Grades 10-12 block.

2. In a separate document on Revision of the Grade 7 Text there is a suggested revision of Chapter 8 - Congruence, Section 8-7, Translations in the Plane, and this is one of the sections proposed for inclusion in Grade 8, Chapter 4, Congruence and Similarity. A check will have to be made with the 7th grade revision to be sure the Grades 7 and 8 sequences on Congruence and Similarity fit together properly with no gaps and a reasonable overlap.

3. Chapter 5, Number Theory in the 1966 Outline, was moved to Grade 7, Chapter 9.

4. In this report the section "Polynomial Algebra", pp. 116, will be of interest to the Grade 8 Writing Team.

Vectors in the 7-9 Outline

Originally in the 1966 outlining session we had concluded that vectors were so important that the idea should be introduced early and developed in spiral fashion throughout the 7-12 curriculum. The statement that Junior High School Science Teachers were in fact using vectors and would use them more if students knew about them, also contributed to our efforts to introduce the idea early. To that end we had included in the outline for Grade 8 (Chapter 3) a treatment of vectors as displacements. This was to be followed in Grade 9 with a chapter which treated operations on vectors, decomposition, connection with analytic geometry, and the use of vectors in geometric proofs.

Actually no detailed outline was prepared for Grade 9, and the Outline for Grade 8, Chapter 3, was lifted from one prepared by Moredock and Sandmann, with some modifications. The Moredock-Sandmann document had been prepared for the junior high school level some time prior to the Summer 1966 outlining session. The 1966 outliners intended for the notion of vectors as displacements to be used in the chapter on Transformations in Grade 9 (Chapter 2 in the 1966 outline).

The various changes proposed this summer (1967) for the 7-9 Outline still retain the idea that vectors should be introduced as displacements and that the chapter introducing them should precede the chapter on Transformations (Grade 9, Chapter 7 in the 1967 outline). However, this timing may be too late to accomplish the objective of introducing the idea early so that it might be used in the Junior High Science Courses.

It also makes it seem unlikely that we will be able to spiral the idea in 7-9 and introduce some more formal algebraic properties of vectors and establish their connection with analytic geometry in the Grade 9 chapter.

This may not be serious. But it will mean that vectors probably should be treated early in the 10th grade block and in a somewhat more systematic way, emphasizing their algebraic properties as well as the geometric idea of a vector as a displacement.

The writers for the 8th grade this year may want to reconsider whether or not the displacements chapter really should be removed from the 8th grade outline as suggested in the changes proposed this summer. If they find that the original reasons for two chapters in successive years in 7-9 are not really compelling, then the spirit and extent of the 9th grade chapter should be rather carefully considered so as to feed in naturally to the 10-11 block which is to use vectors rather heavily in analytic geometry and linear algebra. This is a rather important consideration and should be carefully considered and settled one way or the other, rather than having the subsequent development handicapped by a hasty and precipitous decision at this point.

OUTLINES OF GRADE 8 CHAPTERS

Grade 8 - Chapter 1

Perpendicularity

1966 Outline, pp. 184-191.

Grade 8 - Chapter 2

Coordinate Systems - Distance

1966 Outline, pp. 192-206.

Grade 8 - Chapter 3

Problem Analysis (Strategies)

1966 Outline, pp. 220-253.

1966 Outline, pp. 254-267 Appendix - The Use of Functions in Problem Solving.

Grade 8 - Chapter 4

Congruence and Similarity

This chapter has been put in Grade 8 because of the belief in the spiral development of mathematics. It is also felt that there may be more geometry in Grade 7 than can be comfortably completed there.

It is suggested that this chapter include the following material from the Grade 7 text produced in 1966/67. The Grade 8 writing team should be careful to build on the revised 7th grade text materials.

Grade 7 Chapter 1 - The Structure of Space

1-14. Orientation on a Line
1-15. Orientation in the Plane
1-16. Orientation in Space

Grade 7 Chapter 8 - Congruence

8-7. Translations in the Plane
8-8. Rotations
8-12. Addition and Subtraction Properties for Segments
8-13. Addition and Subtraction Properties for Angles
8-18. Medians of a Triangle
8-19. Angle Bisectors of the Angles of a Triangle
8-20. Altitudes of a Triangle
8-21. Perpendicular Bisectors of the Sides of a Triangle

Grade 7 Chapter 13 - Similarity

13-3. Multiplying Geometrically
13-4. How a Photo Enlarger Works (It is suggested that the title of this section be changed to: The Geometry of a Photo Enlarger)

Grade 9, Chapter 7, Transformations, calls for sections on congruence as an isometric correspondence, and similarity as a ratio-preserving correspondence, so it might be a good idea to introduce this concept here to help get the students ready for the Transformations chapter.

This chapter will include many "originals" involving congruences and similarities. Overlapping triangles and other more complicated originals will be included. One source of challenging problems that might be appropriate in this chapter is Appendix A, Geometry Problems for the Grades 10-12 Block, pp. 215.

This alternate version for Grade 7, Chapter 8, Congruence (Section 8-7 Translations in the Plane) is an example of the style of writing and the type of student activity that many of the group would like to see in the text.

Several comments have been made repeatedly this summer relative to the Grade 7 materials: "It is heavy-handed"; "It is not WST and DIG"; "It tells the students(and often tells well) but there is a lack of involvement and finding the ideas for themselves".

The following sections are an attempt to explicitly display these rather general criticisms. For demonstration purposes, a text section was selected where there was no disagreement with the mathematical content -- the aim of the alternate presentation was to develop the same ideas but from a different approach.

The section chosen was from Grade 7, Chapter 8, on Congruence (Section 8-7 Translations in the Plane). The immediately preceding section has developed the three basic isometry motions of the plane in terms of "slide", "turn", and "flip". The student has worked with many different pairs of congruent figures and pure word descriptions of using these motions to make the figures coincide. The following three sections extend these ideas and give a number sense for description through the use of the coordinate plane.

It is our feeling that the second version presents some exciting potential for classroom use that is not found in the other. It is also our feeling that the second version will develop some self-direction that will make the two following sections develop more naturally. Our proposal is that both methods be tried to see if there is any noticeable difference in student reaction and/or understanding. This could easily be done within the present unit by a simple page-for-page substitution -- that is, it is mechanically feasible.

Note: For the alternate version, the teacher's commentary might contain the celluloid overlay idea to be used as supplement and an idea for discussion ... it might be drawn for the students themselves. Also, the commentary should contain a discussion of how we are using a finite point set to lead to a closed figure to eventually arrive at a translation of the whole plane with the property of congruence preservation. It should be stressed that the pedagogically important idea here is to have the students do the steps and verbalize the ideas -- not just read!

8-7. Translations in the Plane (Present text version)

Motion may be described in many ways. For example, we can say that a man left San Francisco and drove 100 miles north on Route 1. Or we can say that an elevator in an office building ascended from the first floor to the tenth floor. Or we can indicate the motion of a flight in space by describing its orbit.

If we have a wire triangle lying flat on a table, we can slide it on the table in various directions. However, it is difficult to describe a particular motion. In this section we will see how motion may be conveniently described by using a coordinate system.

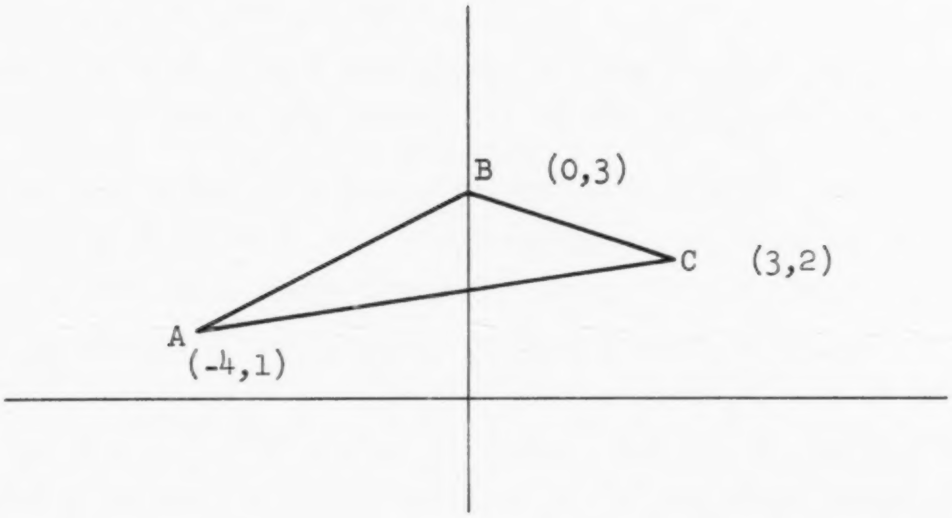


Figure 8-44

Since a point is a definite location in space it makes no sense to talk about moving a point. If we have a different location we have a different point.

Suppose, however, that we have a transparent celluloid overlay which we can place on Figure 8-44. Let us locate points A , B , and C by making pinholes through the celluloid. Now, let us slide the celluloid nine units to the right. Then, the pinhole corresponding to vertex A of $\triangle ABC$ will be at $(-4 + 9, 1)$ or $(5,1)$. We shall call this new point A' . Correspondingly, vertex B' will be at $(0 + 9, 3)$ or $(9,3)$ and vertex C' will be at $(3 + 9, 2)$, or $(12,2)$. In fact, to every point in the plane there will correspond another point determined by the slide of the celluloid nine units to the right. This correspondence is a function. We can describe this function as follows:

> $t : P \rightarrow P'$, where P is any point in the plane, P' is the corresponding point in the plane located by sliding the celluloid nine units to the right.

Or, we can describe this function more compactly, as follows:

$$t : (x,y) \rightarrow (x + 9, y) .$$

We can think of the physical motion of the celluloid as a slide. We refer to the correspondence of points resulting from a slide as a translation.

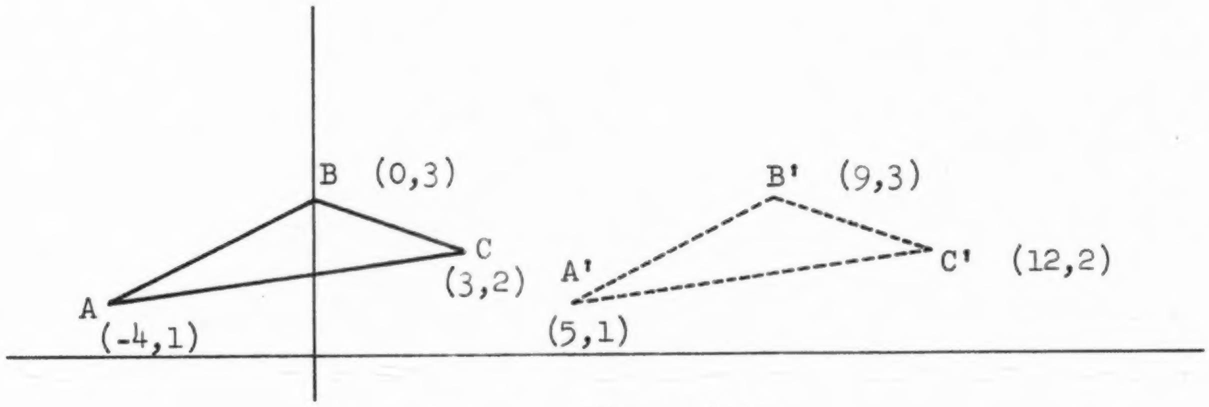


Figure 8-45

In Figure 8-45, $\triangle A'B'C'$ is shown by dash lines. If you draw an arrow from A to A' and another arrow from B to B' the two arrows will have the same length and the same direction. In fact, the arrow drawn from any point in the plane to its image in the plane determined by the function will have the same length and the same direction as any other arrow.

How is $\triangle A'B'C'$ related to $\triangle ABC$? First, we note that $\triangle A'B'C'$ is different from $\triangle ABC$ since the points of $\triangle A'B'C'$ are different from the points of $\triangle ABC$. Second, we surmise that $\triangle A'B'C'$ is congruent to $\triangle ABC$. If we cut out a copy of $\triangle ABC$ and place it on $\triangle A'B'C'$ we will see that the copy will fit exactly on $\triangle A'B'C'$ and we will conclude that our surmise is correct.

It has probably occurred to you that a correspondence may exist between two congruent triangles in the coordinate plane such that one triangle is above or below the other. You would expect that such a correspondence would involve differences between the ordinates of points of the original triangle and the ordinates of points of the new triangle. Let us test our expectation.

Consider $\triangle ABC$, $A(-3,-1)$, $B(1,2)$, $C(7,-4)$, and $\triangle A'B'C'$ which is seven units above $\triangle ABC$. Then the coordinates of the images of points A , B , and C are

$(-3,-1 + 7)$, or $(-3,6)$, for A' ;
$(1,2 + 7)$, or $(1,9)$, for B' ; and
$(7,-4 + 7)$, or $(7,3)$, for C' .

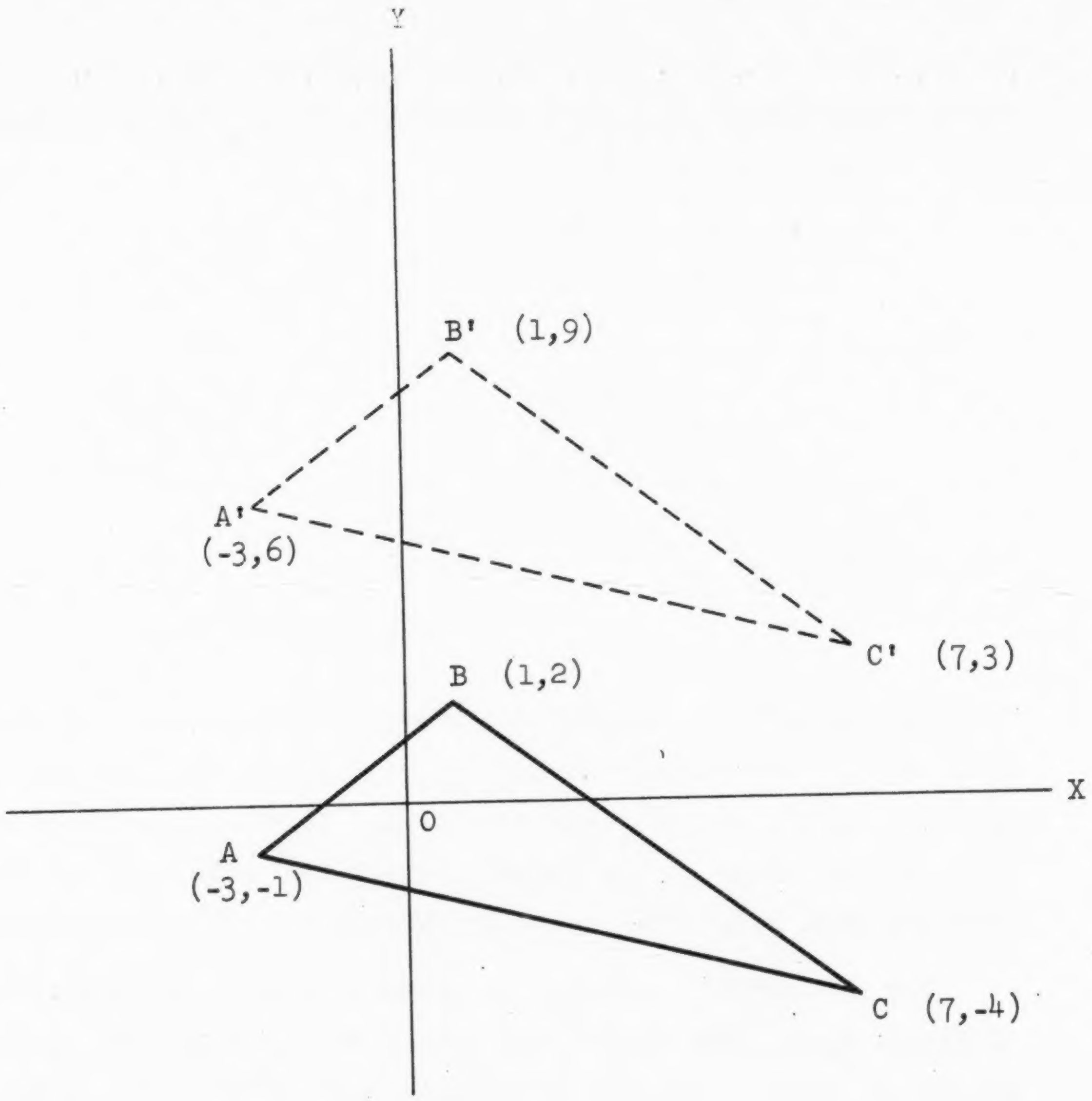


Figure 8-46

The correspondence between the points of $\triangle ABC$ and the points of $\triangle A'B'C'$ is a function. We can describe this function as follows:

$$u : (x,y) \rightarrow (x,y+7) .$$

We can verify the fact that $\triangle ABC \cong \triangle A'B'C'$ by cutting out $\triangle ABC$ and fitting it on $\triangle A'B'C'$.

8-7. Translations in the Plane (Suggested Revision)

The motions we have been discussing can be described precisely if we introduce one other factor: the coordinate plane. Follow the directions carefully, step by step, but see when you can predict the motion.

Exploratory Exercises 8-7a

1. Examine the coordinates of these two points: A(1, 8) , B(6, 8) .

   (a) Do the coordinates name the same points? Why or why not?

   (b) How do the coordinates of A and B compare?

   (c) Plot these points and tell how they are located with respect to each other.

2. Draw a set of coordinates and plot

   A(2, 1) , B(2, 6) , C(3, 3) , D(5, 3) .

3. Draw the closed figure ABCD .

4. Determine new points from the given points by adding 5 to each x-coordinate:

   A(2, 1) → A'(7, 1)
   B(2, 6) → B'(, 6)
   C(3, 3) → C'(8,)
   D(5, 3) → D'(,)

5. Plot the points A' , B' , C' , D' and draw the closed figure A'B'C'D' .

Do the two figures appear to be congruent? What motion could be used to make them coincide? How long is $\overline{AA'}$? $\overline{BB'}$? $\overline{CC'}$? $\overline{DD'}$? How could you describe, in mathematical language, what was done in this problem? (See Figure 8-44)

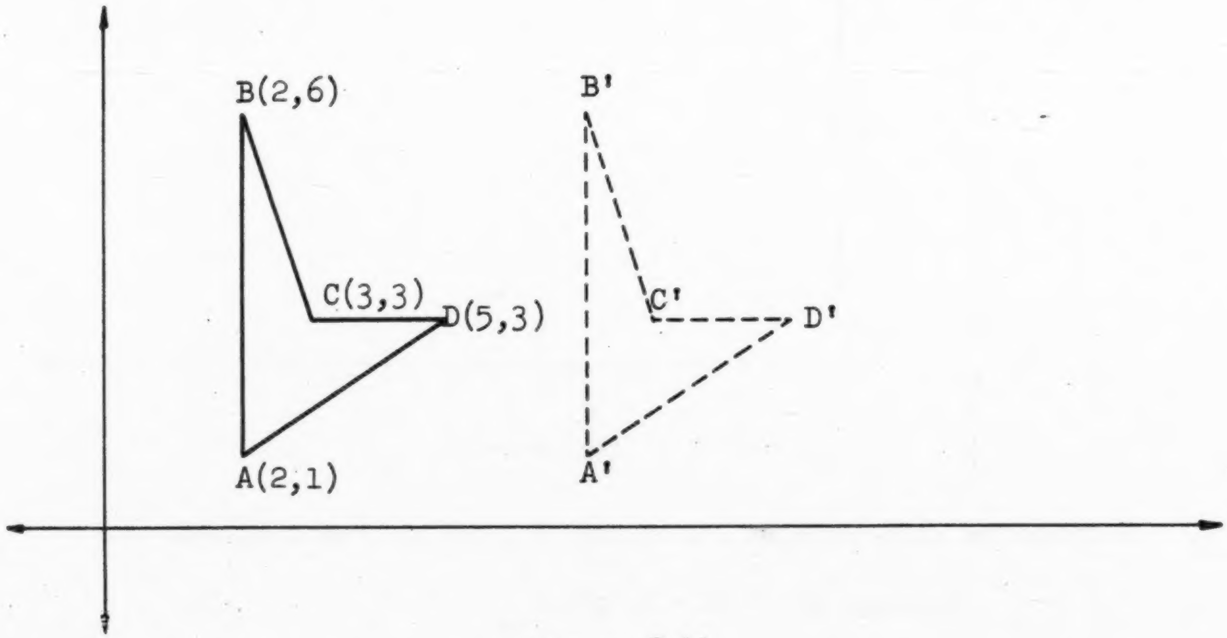


Figure 8-44

If (x, y) are coordinates of a point, P , and a function, t , is defined as

$$t : (x, y) \rightarrow (x + 5, y) ,$$

can you describe the effect of t on the coordinates? What is the effect of t on the point, P ?

6. If A has coordinates (2, 1) , what is the image, A" , of A under t ?

$$A(2, 1) \rightarrow A''(\;,\;) .$$

7. What are the images, B" , C" , D" , of B , C , D , in Exercise 2 under t ?

$B(2, 6) \rightarrow B''(\;,\;)$
$C(3, 3) \rightarrow C''(\;,\;)$
$D(\;,\;) \rightarrow D''(\;,\;) .$

We have applied t to four points, A , B , C , and D . Is the function t restricted to these four points? To how many points of the plane may t be applied? Consider $A(2, 1) \rightarrow A''(7, 1)$ and $B(2, 6) \rightarrow B''(7, 6)$. What happens to the points between A and B under t ? What happens to the segment, $\overline{AB}$?

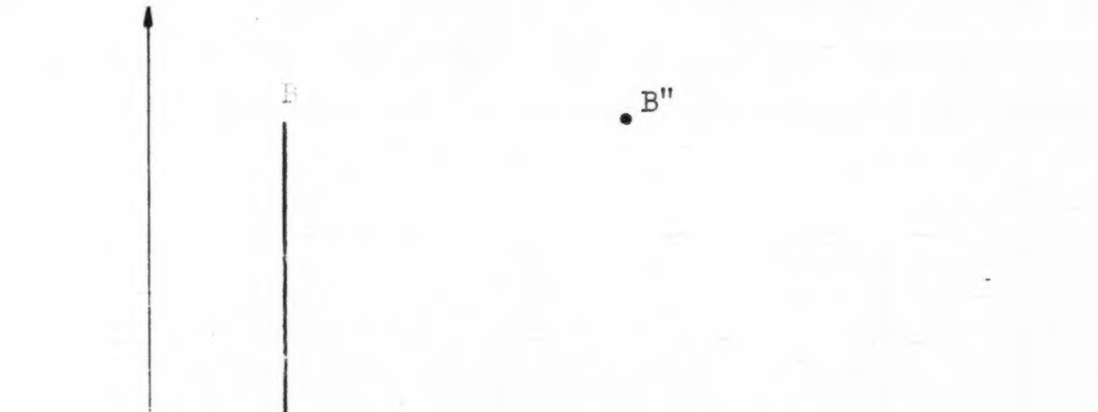


Figure 8-45

8. Suppose we have another function, u , defined as follows:

$$u : (x, y) \rightarrow (x - 5, y) .$$

Is this function the same as t ? If not, how does it differ? Discuss the slide that results from applying u .

We refer to the correspondence of points resulting from a slide with no turning as a translation of the plane. It maps each point of the plane to a new point, and it maps a given figure to a congruent figure. Introducing this function, we can now slide things precisely. But so far we can only translate right and left. What other translations can you suggest?

9. Give a function that you think will slide the given figure 7 units upward.

v : (x, y) → (,   ) .

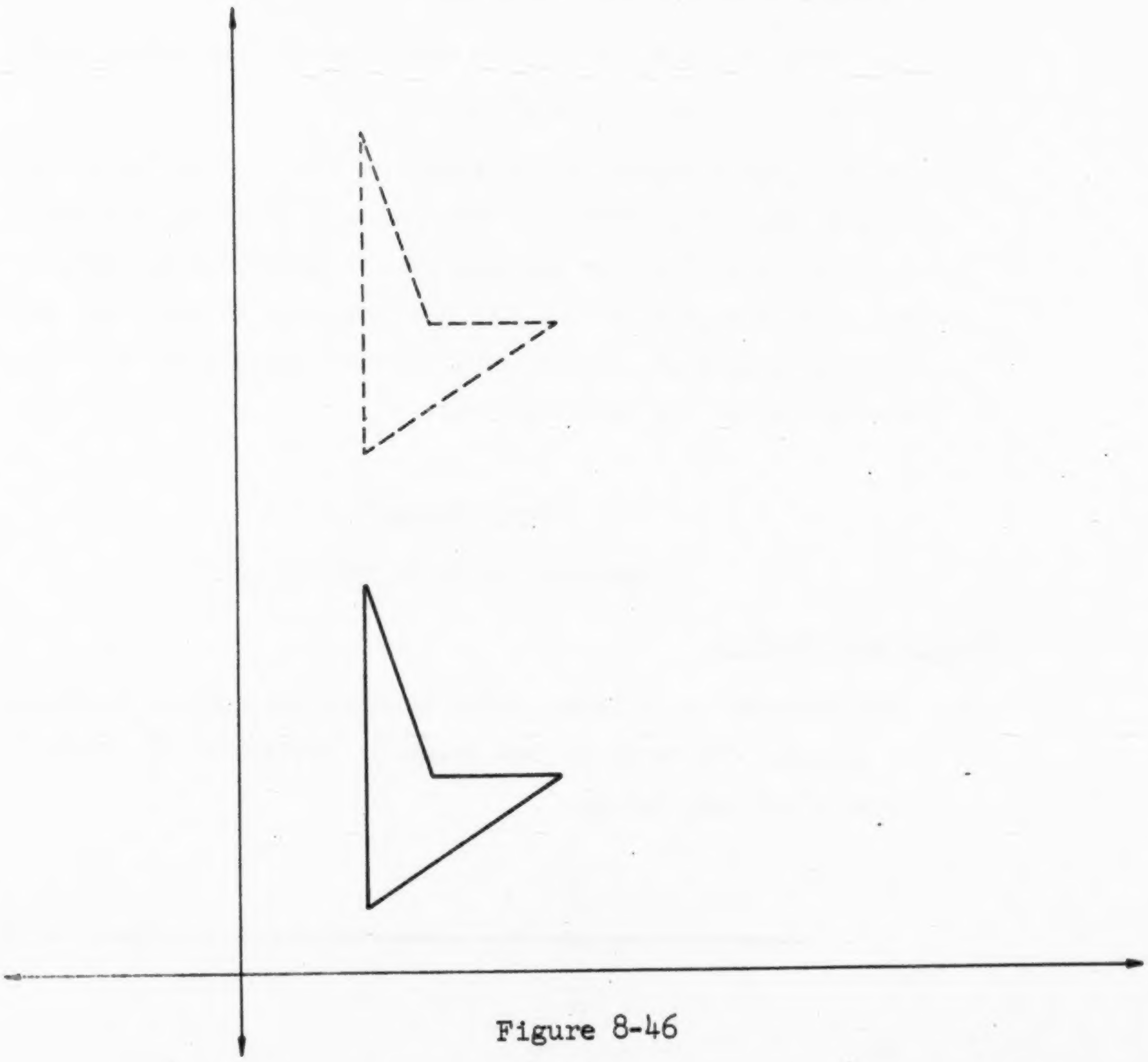


Figure 8-46

Try using your answer on problem 2 and see if you get the expected results.

Now be inventive: give a function that will translate the given figure 10 units to the right and 4 units downward. Try it only if you are uncertain.

W : (x, y) → (,   ) .

Grade 8 - Chapter 5

The Real Numbers Revisited - Radicals

It is suggested that a light treatment of the structure of the various number systems might be included here. In particular some of the material in Grade 7, Chapter 6, Sections 6-1 and 6-2 may be appropriate.

The writing team has three documents to consider in writing this chapter.

(1) The 1966 Outline, pp. 274-279.

(2) Real Numbers, Measure, and Congruence for the Adolescent. This document follows (3) below on pp. 32.

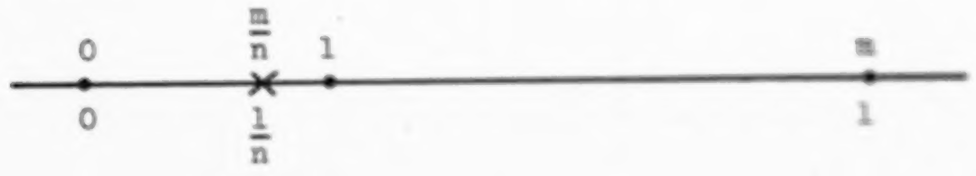
(3) The following suggested revision of the 1966 Outline for this Chapter was submitted, but was not discussed by the group. It is included here for the information of the writing team. The group was much concerned about the presentation of the real numbers, and they desired that the material in this suggested revision be considered by the writing team.

Real Numbers

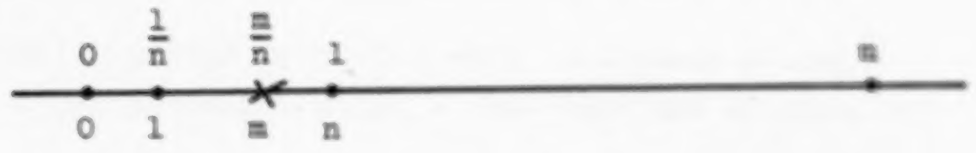
(Suggested Revision of Outline)

1. Points and Numbers

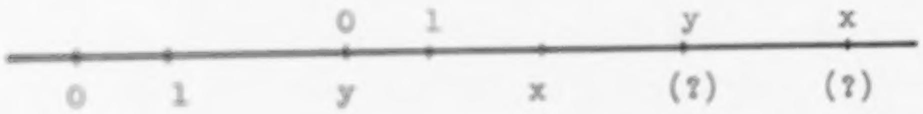
1-1. Coordinates on a line. (For certain purposes a positive ray or a segment are more to the point.) Review of $\frac{m}{n}$ as a location specification, as in



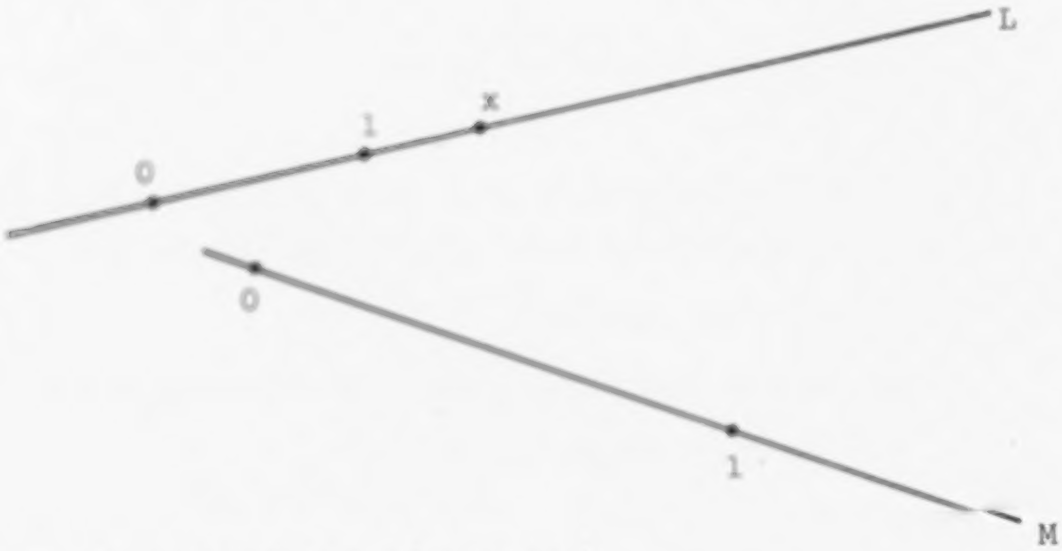
or



Relation of coordinates to congruent segments on a line (or on a pair of lines with congruent unit segments)

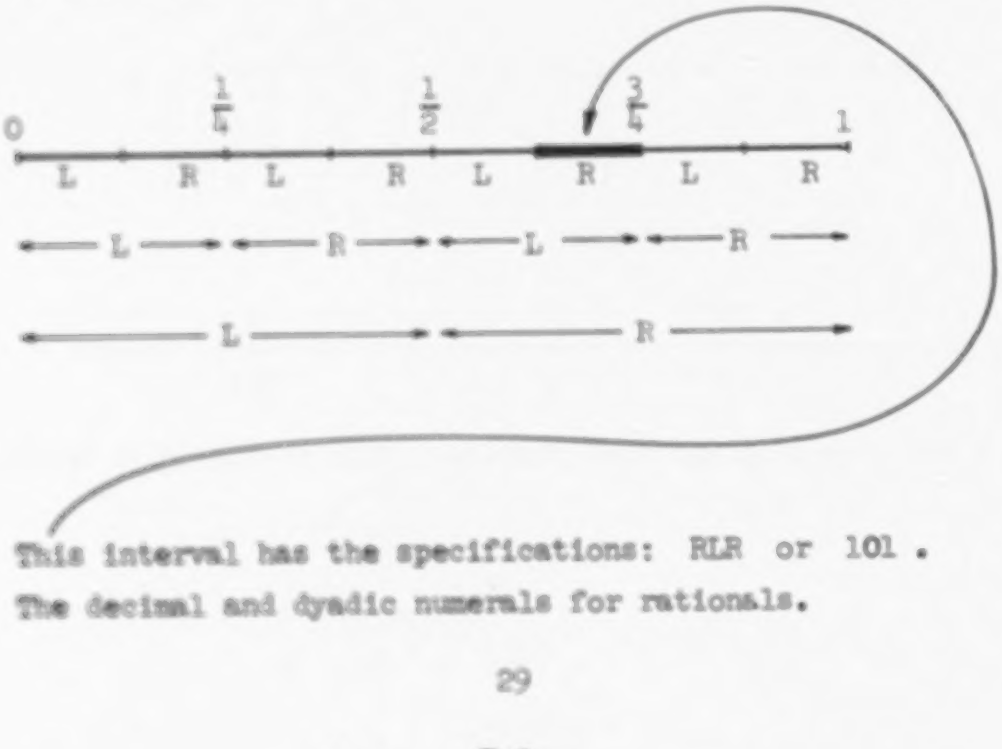


Significance of coordinates on lines with non-congruent unit segments, via similarity. Sample problem: locate the point on M with coordinate x .



1-2. Successive subdivisions of a segment.

Relation of point with coordinate x to that with coordinate [x] to permit restriction to [0,1] . Decimal and dyadic subdivision sequences.


0
$\frac{1}{4}$
$\frac{1}{2}$
$\frac{3}{4}$
1
L R L R L R L R
L R L R
L R

This interval has the specifications: RLR or 101 .
The decimal and dyadic numerals for rationals.

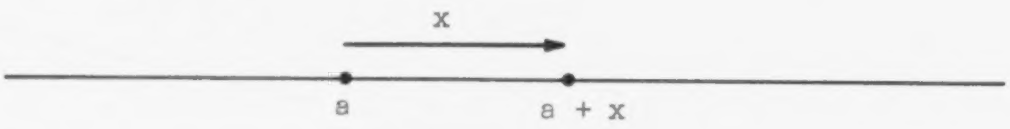
Interpretation of (say) $.\overline{10}$ (dyadic) as an infinite set of specifications for a point; i.e., RLRLRL---- . The nested interval axiom as a working principle.

Exhibition of specifications for irrationals.

Problem: specify an irrational between .01 and .001 ; between $(\frac{1}{2})^{100}$ and $(\frac{1}{2})^{101}$.

2. Geometry of Number Operations

2-1. Displacements on a line and their coordinates

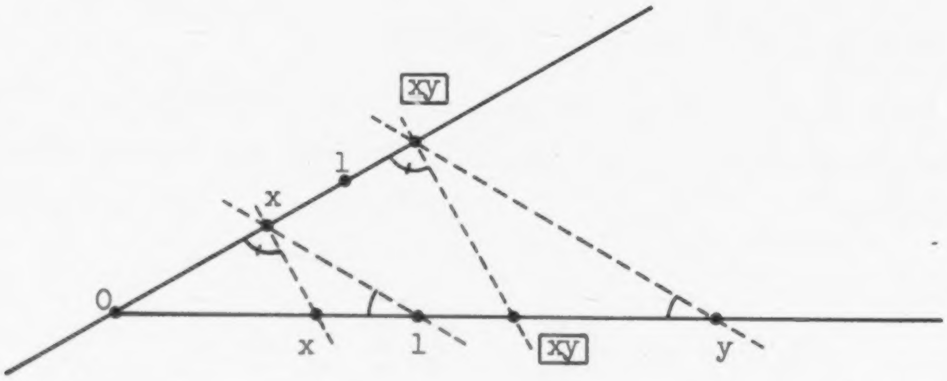


Real number addition defined in terms of displacements; the negative reals and the real opp (opposite) function.

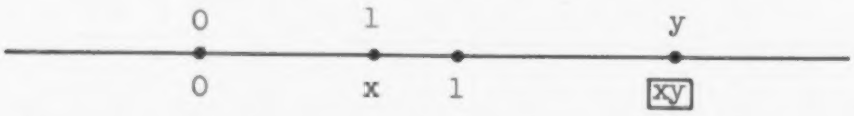
2-2. Products of real numbers.

[Note: Several approaches suggest themselves; for example:

(i) triangle similarity as in



(ii) unit change as in

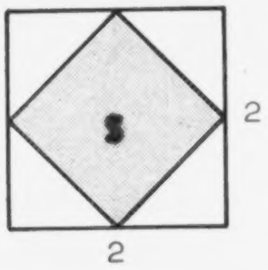


(iii) rectangular area measure. Each rectangle is assigned area $\ell \cdot w$.]

The "betweenness-continuity" property of multiplication, possibly in the following form: $z \to xz$ maps segments of length d onto segments of length less than $n_x d$ for a suitable integer n_x .
The ordered field properties.
[Option: computation problems with decimal or dyadic numerals.]

3. Squares and Square Roots

3-1. Real numbers as measures.
The rectangular area formula. The congruence and partition properties of area. Analysis of the figure



to ascertain the reasoning which leads to $A(s) = 2$.
The Pythagorean theorem reviewed.
Location of $\sqrt{2}$, $\sqrt{5}$, $\sqrt{10}$, etc.
The area of right triangles.

3-2. Square roots. The character of the image of the rationals under the squaring function. Does every positive real number have a square root? [This can be handled by a geometric construction, or a betweenness-continuity argument which can be given algorithmically as, for example:

Let $x_1 = 1$, $z_1 = 3$. Compare $(\frac{3}{2})^2$ with 3 .
Let $x_2 = \frac{3}{2}$, $z_2 = 2$. Compare $(\frac{7}{4})^2$ with 3 .
Let $x_3 = \frac{7}{4}$, $z_3 = \frac{3}{2}$. - - - - -

Now $\bigcap_i [x_i, z_i] = \sqrt{3}$.]

Problems involving numbers generated by square roots over the rationals; as for example:

(i) Which is larger $\frac{\sqrt{3}+1}{2}$ or $\frac{5}{4}$? $\sqrt{3} - \sqrt{2}$ or $\frac{1}{4}$?
$\sqrt{3} + \sqrt{2}$ or $\sqrt{5}$?

(ii) Which of the following are rational and which are irrational; $\frac{(\sqrt{3}+1)}{2}$, $(\sqrt{3}+1)^2$, $\sqrt{12} - \sqrt{3}$, $(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})$?

(iii) The points obtainable from a subdivision of the interval from 0 to $\sqrt{2}$ are those of the form $r\sqrt{2}$ where r is rational. Which of these are rational points? Is $\sqrt{3}$ among them? Is $\frac{1}{\sqrt{2}}$ among them?

The relation between arithmetic and geometric means; area and similarity interpretations of geometric mean.

[Note: Cube roots can easily be added to this chapter.]

The 1967 Outlining Group urges that the following criticisms and suggestions be considered carefully in the revision of Grade 7. The writers of Grade 8 are urged to note these suggestions and criticisms of the treatment of real numbers, measure, and congruence in Grade 7.

Real Numbers, Measure, and Congruence for the Adolescent

A review of several chapters of the seventh grade manuscript leads me to conclude that an adequate position has not been formulated and consistently pursued in regard to the fundamental notions listed in the title. If it is agreed that the treatment in Grade 7 should connect with the stage reached at the end of SMSG Grade 6 and should anticipate the more mature formulations to come, then the current work invites amendment.

Real Numbers. In the elementary school there are no real numbers. It is implicitly suggested in several places that to each segment (or position) on a coordinatized line there corresponds a measure (or coordinate), but there is yet no evidence of incommensurability. The attitude of the seventh grade text may be disputed, but for the most part it appears to suppose that the real numbers are there -- as displacements or congruence classes of segments -- and that properties established for the rationals extend to the reals. Nothing is said, however, to clarify and support this position. More important, the questions relating to the reals are deceptively finessed.

For example, it appears to me that the significance of the Pythagorean theorem (Chapter 14) is blurred and the argument for it contaminated. What is meant by the length of the hypotenuse? More to the point, what is its square and what has that to do with the area of a square on the hypotenuse? A very reasonable conclusion of the argument in its present setting is that the hypotenuse has no length.

Earlier, in Chapter 10, there is an opaque reference to the circumference of a circle as measured by its radius, summarizable as follows:

(i) it is a number;
(ii) it is not a rational number;
(iii) it is therefore expressible by an unending decimal;
(iv) it can be approximated to any degree by rationals.

It seems to me that a foundation sufficient to give meaning to any of these statements has not been laid.

In Chapter 14, which professes to deal with real numbers, the treatment is evasive; e.g., real numbers are not clearly related to points of a line, operations on real numbers are not defined (the field properties are ultimately postulated) and the significance of an infinite decimal is not discussed.

I urge that the rationale which led to this unsatisfactory climax and uncertain foundation be reconsidered.

Measure and Congruence. The attitude toward real numbers has implications for the treatment of measures, as noted above. Congruence is the basic link and the current treatment of these connections raises additional questions.

In elementary school congruence is taken directly from perception and hence is basic. Rational measure values (length, angle, area) are specifiable in terms of congruence. In Chapter 8, however, it is asserted initially without amplification that congruent segments (angles) have the same measure. Later the converse is used without comment. The argument for the congruence of vertical angles includes:

(i) Every angle has a measure and angle sum corresponds to number sum, (unstated);

(ii) vertical angles have the same measure (by subtraction, this is explicit);

(iii) hence vertical angles are congruent.

This attack seems dubious in any context but particularly in a chapter in which measures can, and perhaps should, be avoided altogether.

Again in Chapter 10 (measure) angle measurement is introduced with:

> "Having agreed on a unit of measure we may say that two angles are congruent if they have the same measure."

The rationale for this essentially circular approach is not clear particularly in a pre-real number setting.

All this suggests more thought be given to:

(i) the transition from rational to real measure values, and

(ii) the origin and development of the characteristic properties of measures.

Grade 8 - Chapter 6

Truth Sets of Mathematical Sentences

1966 Outline, pp. 280-281.

Grade 8 - Chapter 7

Quadratic Polynomials as Functions

1966 Outline, pp. 282-286.

Grade 8 - Chapter 8

Probability

This chapter was not outlined in 1966. The following document, Probability and Statistics for Grades 8 and 9, was produced this summer (1967) for the guidance of the writing team. The writing team also has the Grade 7 material that has been produced and must decide how to make the transition to Grade 8.

The group seemed in favor of the revised outline, but there was no strong clear-cut directive.

The document, Probability and Statistics for Grades 8 and 9, in a sense supercedes the next document, Probability and Statistics, Grades 7, 8, 9, 10 or 11. The second document is included here for the information of the writing team; it contains good ideas that should not get lost.

Probability and Statistics for Grades 8 and 9

The authors of this document question the feasibility of the approach toward probability taken in the Outline and already implemented in Grade 7. Our greatest fear is that the student will not see that a probability model is constructed to reflect a physical situation. More important, the student should understand that the assignment of probabilities, while arbitrary, reflects the physical situation as determined by experiment. Why, in a coin tossing game, is the à priori probability for heads chosen to be $\frac{1}{2}$? What arguments could a 7th grader advance to refute the following argument: If two coins are tossed there are 3 outcomes possible (2 heads, 2 tails, or 1 of each) and that hence each should be assigned the probability $\frac{1}{3}$?

We believe it is more natural to approach Probability and Statistics from the standpoint of statistics and an elementary analysis of a collection of data. The following outline suggests a possible treatment for Grades 8 and 9.

(We do not feel sufficiently competent to provide a detailed outline for the high school course in Probability and Statistics.) Our grand goal for Grade 9 is not so much a command of the calculus of probabilities as it is a feeling for the strengths and weaknesses of assertions like:

1. Toothpaste A is significantly more effective in preventing tooth decay than toothpaste B .
2. Lung cancer can be statistically linked with cigarette smoking.
3. The likelihood of rain today is 60 percent.
4. The average 15 year old boy weighs 105 pounds and is 5'10" tall.
5. Should I purchase a car battery for $20 which may last 18 months or one for $30 which may last 30 months?

6. Two radio signals are heard on the same frequency. One is code from a Russian satellite, the other is noise from outer space. How can we identify which is which?

7. A plastic toy manufacturer uses a machine which unfortunately produces defective toys 10 percent of the time. He is considering buying a new machine at the cost of $10,000 which will produce his toy and which is claimed to have a defective rate of only 5 percent. How should he decide if (a) the new machine has a defective rate less than 10 percent, and (b) it is an economical replacement?

GRADE 8.

1. Frequency Distributions.

1-1. Data from observations where the entire population is known. Select examples which can be developed by the students.

(1) Heights of class members.

(2) Distance class members can throw a ball.

(3) Standing broad jump.

(4) Test scores.

(5) Birthdays -- by the month.

(6) Number of children in families of class members.

(7) Age of students in months.

(8) Number of letters in last name. (Also first name. Compare.)

(9) Vowel frequencies in newspapers. Compare English and foreign language papers.

(10) Measurement. With a ruler marked in millimeters, let each student measure a line segment of about a yard.

(11) Estimate midpoint of a line segment of about a foot by eye. Then measure the estimates.

(12) Weight of apples (or oranges) in a box.

1-2. Graphs of data. Grouping of data (give rules of thumb). Continuous model?

1-3. Relative frequency. Cumulative frequency.

Raise lots of questions about the properties of the distributions discussed above.

1-4. Mean, Mode, and Median.

Develop as numbers which describe the total distribution; that is, as examples of number valued functions of the set of distributions.

Give different distributions with the same mean.

Percentiles.

Rescaling: If, for example, in the ball throwing experiment, the distances range from 60' to 150', we could shift the origin so that the range is -45 to 45 or we could rescale so that the range of values is -1 to 1. Compare a shift of the origin with change of scale of the axis. Try out both on examples in Section 1-1. Contrast scaled and unscaled distributions of Examples 2 and 3.

1-5. Variance and Standard Deviation.

Treat as further examples of numbers which describe the whole distribution. Compute for the various distributions in Section 1-1. Ask for comments!

Change of scale effect on variance and standard deviation.

Look to Chebychev's inequality but don't emphasize.

1-6. Subpopulations (Samples).

For example, select out distributions for both boys and girls in the examples in Section 1-1. Plot both distributions on the same graph. Compare. Compare with whole. Compute means and standard deviation. Repeat when, for example, the subpopulation consists of those with first initial A - L and with first initial M - Z.

1-7. Samples.

Treat as similar "hunks". Show how some of the examples in Section 1-6 seem to reflect total distribution while others do not. Compare means and standard deviation. Can these sample statistics be used in prediction?

1-8. Measurement -- Distribution of Errors

Approach from the point of view of comparing different groups of measurements of the same object. Example: Have the class measure with a ruler marked in millimeters a line segment about a yard long. Now consider different subpopulations of different size as though they had determined the length of the segment. Compare. Do not try to suggest that an underlying distribution for the errors in measurement might exist. Just treat it as, "This is what we got".

GRADE 9.

1. Examples from Bernoulli Trials.

Coin tossing, spinners, thumbtacks. (Pick up link with Grade 7.) Perform say 100 trials of coin tossing 20 times. Compute means and standard deviation of these 20 experiments. Repeat the experiment with different sample sizes than 100 . Compute the means and standard deviations -- relate to size of sample.

2. Probability Models.

Take another look at Grade 7. Treat as modeling problem.

Construct Probability space - Event space.

Assignment of probabilities.

Elementary calculation of probabilities.

Give tables for the binomial distribution to avoid complications of sophisticated counting.

3. Elementary Testing of Hypothesis.

The teacher presents statistics from the spins of an unknown spinner. How is the area of the spinner divided?

An ESP experiment with coin tossing: Is the subject doing "significantly" better than he could by guessing? Develop notion of maximum likelihood from the point of view of rejection-acceptance tests. For example, with the spinner problem with a 90 percent confidence we might reject the hypothesis that the distribution was 1/4 - 3/4 and accept that it was 1/2 - 1/2 . We might also accept a great many others and reject a great many. Which seems most favorable?

Go back and pick up examples from Section 1-6. If a student's performance is given can we decide whether the student is male or female? Suppose, for example, we know the distance of the students standing broad jump. One way of reducing the complicated distribution of distances to a Bernoulli trials situation is to make a pairing of the boys and girls. For each pair, record a 1 if the boy's score exceeds that of the girl, and -1 otherwise. Refinements in the method of pairing can bring out other interesting phenomena. For example, is height a key factor in the standing broad jump? Match the pairs so that they have the same height.

4. Problems Requiring more Complex Computations of Probabilities.

Problems where there are more than two outcomes.

ESP experiments where the subject calls the cards from a deck of say 12 cards. When is a long run of successful guesses significant?

Dice (or a suitable euphemism).

Independence of trials.

Selection without replacement.

(In this section we would continue the emphasis to a test of hypotheses. These experiments will lead to the next section where the counting tools are developed.)

5. Combinatorics -- Systematic Counting.

Inclusion-exclusion principle.
Multinomial coefficients.
Tree diagrams.

6. Conditional Probabilities.

Independence of tests.

7. Random Variables.

Return to Grade 8 type of examples and read off various functions of the distributions. Use Bernoulli trials. Discuss the R. V. which is the number of tosses before the first head.

Expected value. Expected value of a sum of R. V. is the sum of the expected values. Use this as an aid in determining probabilities and in the combinatorics of Section 2.

8. Correlations Between Different Random Variables.

Curve fitting -- Distinction between best fit and goodness of fit.

This document, in a sense, is superceded by the previous document, Probability and Statistics for Grades 8 and 9. It is included here for the information of the writing team; it contains good ideas that should not get lost.

Probability and Statistics, Grades 7, 8, 9, 10 or 11

GRADE 7. (1966-67 Version)

1. Assigning Probabilities to Outcomes.
   - 1-1. Bases for equally likely assignment fair games.
   - 1-2. Probability assignments based on the equally likely case; spinners, marbles in a jar, etc.
2. Pairs of Trials.
   - 2-1. Tree diagrams, counting outcomes.
   - 2-2. Assigning probabilities, equally likely assumption.
   - 2-3. Examples of extensions to 3 and 4 trials.
3. Inferences from Relative Frequencies.
   - 3-1. Spinner trials with sizes of sectors unknown.
   - 3-2. Thumb tack throws.
   - 3-3. Data from treatments, production lines, etc.
4. Unions of Events.
   - 4-1. Relation of $P(A \cup B)$, $P(A)$, $P(B)$, $P(A \cap B)$.
   - 4-2. Mutually exclusive case.
5. Intersections of Events.
   - 5-1. Pairs of trials with and without replacement.
   - 5-2. Independent events.

GRADE 8.

1. Frequency Distributions.
   - 1-1. Data which can be regarded as the outcomes of an "experiment"; frequency, relative frequency and cumulative frequency distributions

and their graphical presentation; examples such as: class absences during the year by day of the week, distribution of birthdays by month, number of children per family, age of pupils in months, occurrences of each vowel in several lines of text.

1-2. Means and Percentiles.

The mean as representative of a distribution, significance of $s = n\bar{x}$; means in comparisons, are girls in class older than boys? Computation of means by change in scale. Median, percentile points in a distribution, standard arithmetic test scores, problem of dividing class into 3 or 4 groups by height, etc.

2. Probability Distributions.

2-1. Probabilities as "weights" associated with long run relative frequencies. Outcome probabilities in selection from known finite populations (as in 1-1 above).

2-2. Means of sample distributions as estimates of population means, problem of accuracy of estimates of population means.

2-3. Probabilities in coin or spinner types of situations; problem of testing an hypothesis as, e.g., in pupil taking true-false test, ESP experiment with cards or coin, Friday or Monday absences from school, etc.; testing hypothesis requires deriving probabilities from certain assumed probabilities.

GRADE 9.

3. Probabilities in Repeated Trials.

3-1. Implications of independence of trials, the problem of counting the number of k-element subsets of an n-element set.

3-2. Selection without replacement; the problem of counting sequences.

4. Methods of Counting Possibilities.

4-1. Tree diagrams, the fundamental counting principle.

4-2. The number of k-sequences and distinct k-sequences from an n-element set. The number of k-element subsets; relation to binomial theorem.

5. Testing Hypotheses.

5-1. Test hypothesis $p = \frac{1}{2}$ in several specific examples, e.g., sex, births in March-September, using central intervals; apply similar technique to a case with $p \neq \frac{1}{2}$; have class devise tests, e.g., of independence of trials in births.

5-2. Conduct ESP test using order in shuffled deck of 10 or 12 cards.

6. Standard Deviation.

6-1. Binomial distribution with $p = \frac{1}{2}$; relativized forms and differences in spread. Standard deviation (s.d.) as a natural unit in measuring spread, standard deviation of a frequency distribution.

6-2. The (binomial) law of large numbers.
Design of experiments, e.g., distinguish between $p = .5$ or $p = .6$ as appropriate probability assignment.

GRADE 10 or GRADE 11.

7. Conditional Probability.

7-1. The Mendelian model; 3-spinner or urn simulation of genotypes; problems in population genetics, e.g., all AA are wiped out or AA is lethal; Hardy's Theorem.

7-2. A problem in sex linked genetics, e.g., hemophilia, color blindness; comparison with data.

8. Markov Chains.

8-1. Genotype change over generations; population movements; random walk representation; gambling problems; transition probability matrices.

8-2. Long run trends in Markov processes; matrix multiplication.

9. The Normal Distribution.

9-1. Normalized binomial distributions; conditions under which the normal distribution (as a tabulated function) is a reasonable approximation. Situations in which distributions are approximately normal.

9-2. Common uses of the normal distribution; standardizing test scores; inferences from sample mean and s.d. (standard deviation), repeated selections from normally distributed population; testing randomness of samples; e.g., among 30 honor students 19 are girls (compare with binomial computation), or the class as a sample of all students in the same grade with respect to a normal achievement test.

Grade 8 - Chapter 9

Parallels and Perpendiculars

1966 Outline, pp. 287-298.

In addition to the material suggested in the 1966 outline it is suggested that the following material from Grade 7 be included. This is suggested to lighten the geometry load in Grade 7 and spread the geometry material over a longer period of time.

Grade 7. Chapter 1 - The Structure of Space.

1-12. Curves, Simple Closed Curve
1-13. Convexity

Grade 7. Chapter 12 - Parallelism.

12-4. Parallels and the Circumference of the Earth
12-7. More About Parallelograms
12-8. Families and Networks of Parallel Lines
12-9. Dividing Segments Into Congruent Parts
12-10. Triangles and Families of Parallel Lines
12-11. Networks and Coordinates

In discussing the above proposed shifts of material from Grade 7 to Grade 8, it was pointed out that Sections 1-12 and 1-13 were about the only things that might be new in Chapter 1, so in the Revision of Grade 7 these two sections might be retained.

It was also pointed out that Section 12-4 might well be retained in Grade 7 because it is a nice application of material that has just been taught. Here again the writing team must build on the revised version of the Grade 7 text. Of course, the revised version will not be available for the first writing of Grade 8, so an educated guess will be required here.

Grade 8 - Chapter 10

Properties and Mensuration of Geometric Figures
(Review and Summary)

1966 Outline, pp. 299-311.

There are three more documents that may be useful here and they will follow in this order:

(1) A treatment of Area, Volume, Work and Falling Body Problems Without Limit Processes. This treatment does not use function notation.

(2) A document with the same title as (1) but this treatment uses function notation.

(3) A "Circular" Unit of Measure for Circular Areas.

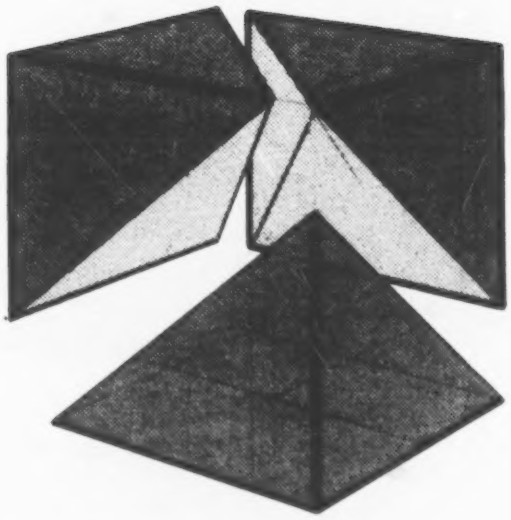
A Treatment of Area, Volume, Work and Falling Body Problems without Limit Processes

It is possible to set up double inequalities for area, volume, falling body and work problems and to solve them using only the simple algebra of inequalities. This fact may prove useful in giving significant applications of algebra at an early level.

It is thought that some of this material may be suitable for use in the treatment of measurement in Grades 8 and 9. In particular, it may be used in Grade 8, Chapter 10, Properties and Mensuration of Geometric Figures; Grade 9, Chapter 4, Measure Functions and Their Properties; and Grade 9, Chapter 10, Measure.

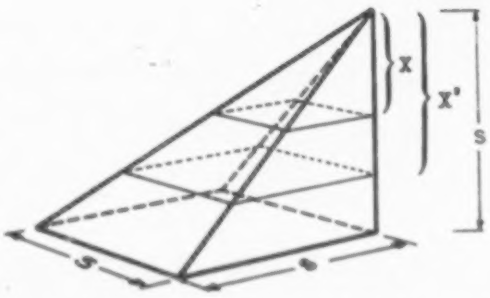
This treatment does not involve function notation. Since function notation has been used from Grade 7 the writers may want to consider using some of the material that is precisely the same except for the use of function notation. This material follows immediately after the treatment not using function notation, pp. 52.

On Volumes and Areas



1. Pyramids, Cones and Spheres.

(a) Consider a pyramid with altitude equal to the side s of the square base and suppose that the vertex is above one corner of the base. By constructing two copies of such a pyramid and putting all three pyramids together we can form a cube of side s and volume s^3. The volume of the given pyramid is therefore $\frac{1}{3} s^3$.



The volume of the pyramid above the plane at distance x below the vertex is

$$V = \frac{1}{3} x^3 .$$

The volume above the plane at distance x' $(x' > x)$ is

$$V' = \frac{1}{3} x'^3 .$$

Then the volume of the frustum between these two planes is

$$V' - V = \frac{1}{3} x'^3 - \frac{1}{3} x^3 .$$

This volume is greater than the smallest cross-sectional area x^2 times the thickness $x' - x$ and less than the largest cross-sectional area x'^2 times $x' - x$. That is,

$$(1) \quad x^2(x' - x) < \frac{1}{3} x'^3 - \frac{1}{3} x^3 < x'^2(x' - x) .$$

This is easy to verify algebraically since

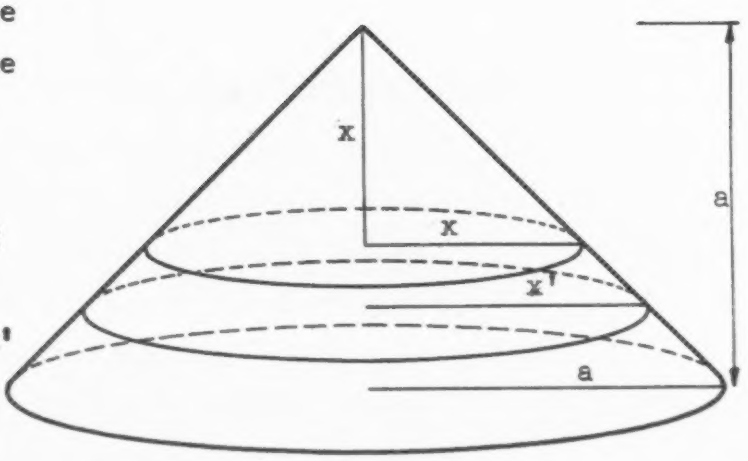
$$\frac{1}{3} x'^3 - \frac{1}{3} x^3 = \frac{1}{3}(x' - x)(x^2 + xx' + x'^2) .$$

It may be shown (see Section 2) that if we require that

$$x^2(x' - x) < V' - V < x'^2(x' - x)$$

for all $0 \leq x < x'$, there is no other possibility than $V = \frac{1}{3} x^3$ (and $V' = \frac{1}{3} x'^3$).

(b) Now take a right circular cone with altitude a equal to the radius of the base. What is its volume?



A section at distance x from the vertex has the area πx^2 while one at distance x' has the area $\pi x'^2$. If V is the volume of the smaller cone and V' the volume of the larger cone, the frustum has the volume $V' - V$. Then we require that

(2) $\pi x^2(x' - x) < V' - V < \pi x'^2(x' - x)$.

What then is V ? If we multiply (1) by π we obtain

(3) $\pi x^2(x' - x) < \frac{\pi}{3} x'^3 - \frac{\pi}{3} x^3 < \pi x'^2(x' - x)$.

Comparing (2) and (3) we find

$$V = \frac{\pi}{3} x^3 \quad (\text{and} \quad V' = \frac{\pi}{3} x^3).$$

Hence the volume of the given cone is $\frac{\pi}{3} a^3$.

If more generally the radius of the base is ka , the radii of the cross-sections at x and x' are kx and kx' respectively. Then (2) is replaced by

(4) $\pi k^2 x^2(x' - x) < V' - V < \pi k^2 x'^2(x' - x)$.

If we multiply (3) by k^2 we find

$$\pi k^2 x^2(x' - x) < \frac{\pi k^2 x'^3}{3} - \frac{\pi k^2 x^3}{3} < \pi k^2 x'^2(x' - x)$$

and so $V = \frac{\pi k^2 x^3}{3}$ $(V' = \frac{\pi k^2 x'^3}{3}$).

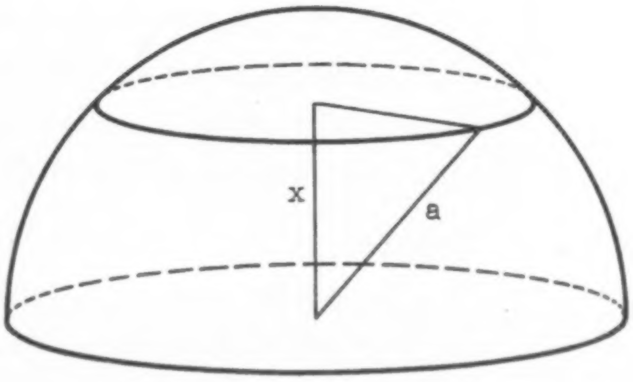
The cone has the volume

$$\pi \frac{k^2 a^3}{3} = \frac{\pi}{3} k^2 a^2 \cdot a = \frac{1}{3} \pi r^2 \cdot a$$

($\frac{1}{3}$ area of base x altitude) .

(c) We turn now to the volume of a hemisphere of radius a . If V is the volume above the base and below a plane at height x and V' the volume below the plane at height x' , the volume between the planes is

$$V' - V .$$



The cross-sectional areas at x and x' are respectively $\pi(a^2 - x^2)$ and $\pi(a^2 - x'^2)$. We therefore require that

$$(5) \quad \pi(a^2 - x'^2)(x' - x) < V' - V < \pi(a^2 - x^2)(x' - x) .$$

From (3), reversing signs,

$$-\pi x'^2(x' - x) < -\frac{\pi x'^3}{3} + \frac{\pi x^3}{3} < -\pi x^2(x' - x) .$$

Adding $\pi a^2(x' - x)$ throughout

$$\pi(a^2 - x'^2)(x' - x) < (\pi a^2 x' - \frac{\pi x'^3}{3}) - (\pi a^2 x - \frac{\pi x^3}{3})$$
$$< \pi(a^2 - x^2)(x' - x) .$$

Comparing with (5) we see that

$$V = \pi a^2 x - \frac{\pi x^3}{3} .$$

Substituting x = a gives $\frac{2\pi a^3}{3}$ for the volume of the hemisphere and hence $\frac{4\pi a^3}{3}$ for the volume of the sphere.

2. Uniqueness.

The method that we have used depends upon the double inequality

$$(6) \qquad S(x' - x) < V' - V < S'(x' - x)$$

where S and S' are the cross-sectional areas at x and x' and V and V' the volumes of the solids bounded by a base plane and the planes at x and x' . (If the cross-sectional area decreases as x increases, S and S' must be interchanged.) We require that (6) shall hold for all $0 \leq x < x'$. From our knowledge of (1) we have been able to find V and V' in some other cases using the simple algebra of inequalities. The question arises: are the answers we have found the only possible ones? The reply is "yes" so long as the cross-sectional areas increase (or decrease) steadily with x .

Proof:

Suppose that we slice our solid with equally-spaced planes perpendicular to the line along which x is measured. We assume for definiteness that the cross-sectional area S increases with x .

If V_1 is the volume of the first slice

$$S_0 t < V_1 < S_1 t$$

where S_0 is the value of S at $x = 0$ and S_1 the value of S at $x = t$, where t is the thickness of the slice.

For the second slice

$$S_1 t < V_2 - V_1 < S_2 t$$

$$\cdots$$

and finally

$$S_{n-1} t < V - V_{n-1} < S_n t \ .$$

Adding

$$(S_0 + S_1 + \dots + S_{n-1})t < V < (S_1 + S_2 + \dots S_n)t \ .$$

The difference between the upper and lower sums is

$$(S_n - S_0)t < S_n t = Mt$$

since S_n is the maximum cross-sectional area M . V of course is the required volume. If there could be a second number $\overline{V}$ satisfying the double inequalities, we would have

$$|\overline{V} - V| < Mt \ . \tag{7}$$

The thickness t of each slice is the distance, d , between the bases of the solid, divided by the number of slices n . This means that (7) becomes

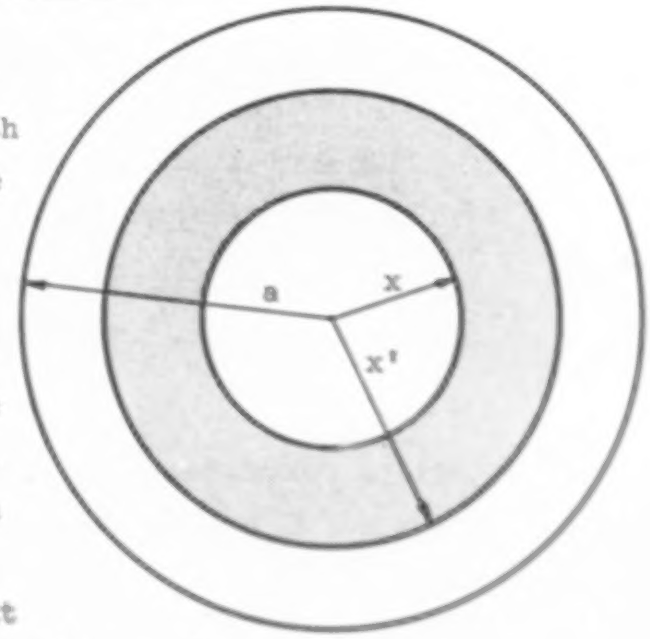
$$|\overline{V} - V| < \frac{Md}{n} \ .$$

By choosing n large enough we have an obvious contradiction. Therefore there can be only one number V which could represent the volume.

The geometrical meaning of (7) is the following. When the solid is sliced into n pieces of equal thickness, the volume is determined within an amount equal to the volume of the largest slice. Since this largest slice can be made as thin as we please, the volume is precisely nailed down.

3. Remarks on the Areas of Circles and Spheres.

(a) The double inequality method leads to the area of a circle if we are willing to start with $C = 2\pi x$ for the circumference of a circle of radius x.



Let A be the area of the circle of radius x, and A' the area of the concentric circle of radius $x'(x' > x)$. Then the annulus or ring shown shaded has the area $A' - A$. It seems intuitively clear that

$$2\pi x(x' - x) < A' - A < 2\pi x'(x' - x) .$$

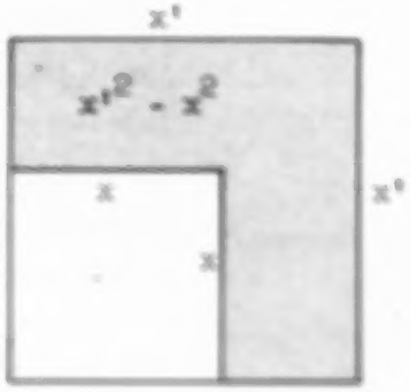
Now

$$(8) \qquad 2x(x' - x) < x'^2 - x^2 < 2x'(x' - x)$$

as we see algebraically since

$$x'^2 - x^2 = (x + x')(x' - x)$$

and geometrically from the following figure.



Multiplying (1) by π

$$2\pi x(x' - x) < \pi x'^2 - \pi x^2 < 2\pi x'(x' - x)$$

which leads to $A = \pi x^2$.

(b) The surface area of a sphere may be treated similarly. If $S = kx^2$ is the area of a sphere of radius x and $S' = k\,x'^2$ is the area of a sphere of radius x', we have for the volume $V' - V$ between them

$$kx^2(x' - x) < V' - V < kx'^2(x' - x) .$$

However, we know V and V' so that we may write

$$kx^2(x' - x) < \frac{4\pi}{3} x'^3 - \frac{4\pi}{3} x^3 < kx'^2(x' - x) .$$

We need to find k .

Since from Section 1

$$(1) \qquad x^2(x' - x) < \frac{x'^3}{3} - \frac{x^3}{3} < x'^2(x' - x) ,$$

$$4\pi x^2(x' - x) < \frac{4\pi x'^3}{3} - \frac{4\pi x^3}{3} < 4\pi\, x'^2(x' - x) .$$

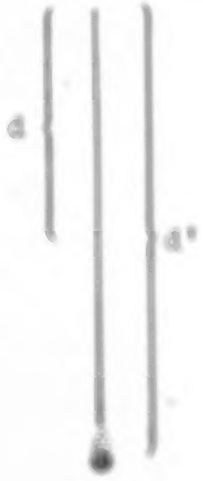
Hence $k = 4\pi$ and $S = 4\pi x^2$. The surface of a sphere of radius $\underline{a}$ has the area $4\pi a^2$.

4. Galileo assumed that the speed v of a body dropped from rest is a constant k times the elapsed time x ,

$$v = kx .$$

What is the distance d fallen in time x ? The distance covered between the time x and the later time x' is surely

$$d' - d .$$



This is greater than the distance that would have been covered had the speed remained equal to kx . Throughout the time interval $(x' - x)$ and less than the distance which would correspond to the constant speed kx' .

Hence

$$(2) \qquad kx(x' - x) < d' - d < kx'(x' - x) .$$

The double inequality (2) would arise in finding the area A under the graph $y = kx$ and above $[0,x]$. This area is $\frac{1}{2} \cdot x \cdot kx$. Hence $d = \frac{kx^2}{2}$.

5. Hooke proved that the force exerted by a spring is kx where x is the extension from the unstretched position. If W is the work required to stretch the spring the distance x , the work required to stretch it from x to x' is surely $W' - W$. Then we see immediately that

$$kx(x' - x) < W' - W < kx'(x' - x) .$$

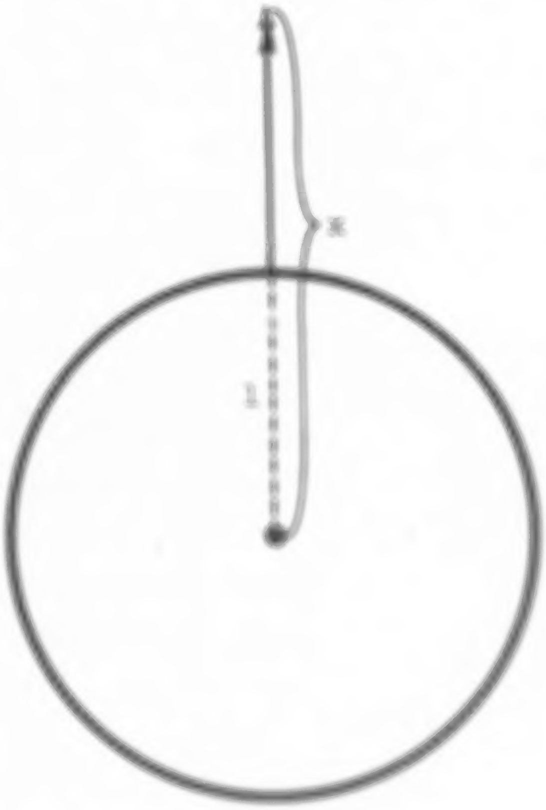
The solution is $W = \frac{kx^2}{2}$. Same mathematical model for a different physical situation!

6. A more interesting problem concerns the work (energy) required to lift a rocket 4000 miles above the earth's surface. Use the earth's radius as a unit of length. The force at the distance x from the center of the earth is $\frac{k}{x^2}$ where k is the weight of the rocket when $x = 1$. If W is the work to go from 1 to $x (x > 1)$ we require that

$$\frac{k}{x'^2}(x' - x) < W' - W < \frac{k}{x^2}(x' - x) .$$

Since $\frac{1}{x'^2} < \frac{1}{xx'} < \frac{1}{x^2}$

$$\frac{k}{x'^2}(x' - x) < \frac{k}{xx'}(x' - x) < \frac{k}{x^2}(x' - x) .$$



Because of uniqueness we can set

$$W' - W = \frac{k(x' - x)}{xx'} = \frac{k}{x} - \frac{k}{x'} .$$

At $x = 1$, $W = 0$. So

$$W' = k - \frac{k}{x'} .$$

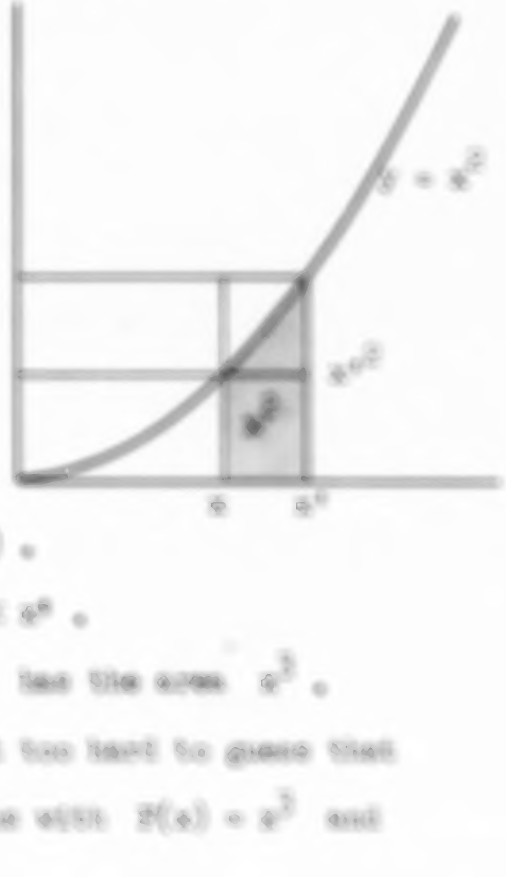
When $x = 2$, $W' = k - \frac{k}{2} = \frac{k}{2}$. If k is in pounds and we wish the answer in ft.-lbs. we should multiply by 4000×5280 .

Some of the outlining group felt that since function notation is familiar, this treatment may be preferable over the previous treatment not involving function notation.

A Treatment of Area, Volume, Work and Falling Body Problems without Limit Processes Using Function Notation

It is possible to set up double inequalities for area, volume, falling body and work problems and to solve them using only the simple algebra of inequalities. This fact may prove useful in giving significant applications of algebra at an early level. The following sketch shows the essential idea.

1. Let $F(x)$ be the area under the parabola $y = x^2$ above the interval $[0, x]$ and $F(x^*)$ the area above $[0, x^*]$, $x^* > x$. Then the area above $[x, x^*]$ is
$F(x^*) - F(x)$
which is clearly between the areas of two rectangles with base $x^* - x$. Then



(1) $x^2(x^* - x) < F(x^*) - F(x) < x^{*2}(x^* - x)$.

This double inequality holds for all $0 \leq x < x^*$.

The rectangle with base x and altitude x^2 has the area x^3. Clearly $F(x)$ is less than $\frac{1}{2}x^3$. It is not too hard to guess that $F(x) = \frac{1}{3}x^3$. However, let us see what happens with $F(x) = x^3$ and $F(x^*) = x^{*3}$. Since

$$\begin{aligned} x^{*3} - x^3 &= (x^* - x)(x^2 + xx^* + x^{*2}) \\ &< (x^* - x)\,3x^{*2} \end{aligned}$$

and

$$> (x^* - x)3x^2$$

we have

$$3x^2(x^* - x) < x^{*3} - x^3 < 3x^{*2}(x^* - x)$$

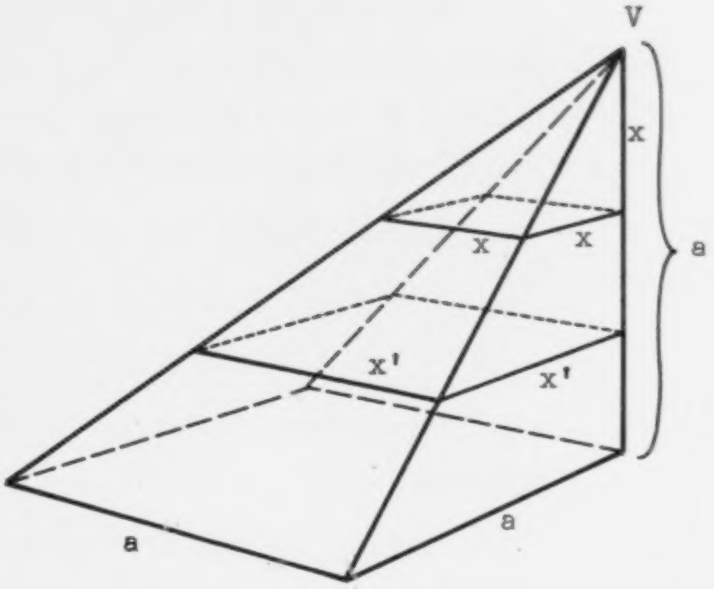
which differs from (1) by the presence of the 3's . Dividing we obtain

$$x^2(x' - x) < \frac{x'^3}{3} - \frac{x^3}{3} < x'^2(x' - x)$$

so that $F(x) = \frac{x^3}{3}$ is a solution of (1) for all $0 \le x < x'$.

It may be shown (proof on request) that this is the only solution of (1) for which $F(0) = 0$.

2. To find the volume of the square pyramid shown, let $F(x)$ be the volume between the vertex and the plane at distance x below it. Then $F(x') - F(x)$ is the volume between the planes at x and x' . This volume is greater than the thickness $(x' - x)$ of the slab times the smallest cross-sectional area x^2 and less than $(x' - x)$ times the largest cross-sectional area x'^2 . Then



$$x^2(x' - x) < F(x') - F(x) < x'^2(x' - x) .$$

This is the same as (1), so that $F(x) = \frac{x^3}{3}$. The result can be verified immediately since the pyramid above the x-plane is $\frac{1}{3}$ of a cube of side x .

The volumes of cones and spheres are easily found similarly, assuming πr^2 for the area of a circle.

A "Circular" Unit of Measure for Circular Areas

Traditional measures of area are usually associated with square areal units which, with circular areas, usually bring in dat old debbil π. Electricians have sometimes used circular units:

1 circular mil = the area of a circle whose diameter is 1 mil, or .001 inches.

With this unit certain computations involving areas (and electrical resistances) become easier; thus the area of a circle of diameter .004 is simply $4^2 = 16$ c.m. and so on. A source of situations for modeling is also easily available since the resistance varies inversely as the area . . .

Another useful unit of volume can come from the same source:

1 mil - foot = the volume of a cylinder of diameter .001 inch and length 1 foot.

This volume unit would also rationalize many problems in volumes of cylinders.

Perhaps the use of these units, since they lead from rational inputs to rational outputs, would allow pupils to master the conceptual material more readily, before they have to cope with the irrationalities of π and their teachers.

Grade 8 - Chapter 11

Spatial Perception and Locus

1966 Outline, pp. 312-315.

Grade 8 - Chapter 12

Systems of Sentences in Two Variables

Since systems involving inequalities as well as equations will be studied, it was felt that the title should be changed -- using the word Sentences instead of the word Equations in the title.

See the 1966 Outline, pp. 316-319.

The following is a revision of the 1966 Outline, using new material that was written this summer. It will be seen that this new outline essentially covers the 1966 Outline and a little more.

Revision of 1966 Outline

Background: Graph of Linear equations.
Solution sets of equations and inequalities.

Purpose:

1. To review the graphical representation of equations and inequalities in one variable.
2. To develop the graphical representation of solution sets of systems of first degree sentences in two variables.
3. To formulate the concept of equivalent systems and introduce the method of linear combinations for arriving at algebraic solutions.
4. To examine various cases of systems of equations and their graphical interpretation -- inconsistent, consistent, dependent; parallel, coinciding, and intersecting lines.
5. Extend work with systems of inequalities to general linear inequalities and to regions bounded by several straight lines, in preparation for finding convex regions in elementary linear programming problems.

Section 12-1: A Decision Problem. (Class Involvement)
(See Appendix C, Section 1)

"Suppose you are president of a division of a large corporation called 'General Engines'. Your division manufactures one make of car and one make of truck. How many cars and trucks should be scheduled for the next year's production to make as large a profit as possible?"

(Almost all of the pertinent information should be contained in the T.C. to be released by the teacher as the need arises in the discussion. See Appendix C, Section 1.)

Additional Information:

1. The profit on each truck is different than the profit on each car.
2. The supply of steel is limited.
3. It takes more steel to build a truck than a car.
4. The factory's capacity to produce units of cars and trucks is limited.

1.1 Class Exercises

1. What do you know about profit? On each car? On each truck?
2. How much steel is available?
3. How much steel does it take to make a car? a truck?
4. How many cars can you build? What's your total profit?
5. How many trucks can you build? What's your total profit?
6. What's your decision?
7. Is there any information that you haven't used yet?
8. If you make only cars, do you use all the steel?
9. If you make only trucks, do you use all the steel?
10. If you make only cars, can the factory make that many in a year?
11. Can the production be only trucks, and keep the factory operating at full capacity? Why, or why not?

Suppose we know that this year, 90,000 trucks and 410,000 cars were made;

12. Discuss the total profit on the cars and the trucks, assuming they were all sold.
13. Discuss the amount of steel used to do the above.
14. If 15,000 tons of steel will be left over from this year's production, how many tons will be surplus if you stick to this year's production plan?

Copy and fill in the following table showing the information you have collected.

	Number of Cars	Number of Trucks	Unused Steel	Unused Capacity	Total Profit
1	500,000				
2	0				
3	450,000				

Inspecting the table you see that decreasing the number of cars produced from 500,000 to 410,000 (and producing trucks instead) decreases the amount of unused steel and increases profits. However, if you continue to decrease the number of cars, say to zero, then profits decrease too. This suggests that somewhere in between 410,000 cars and zero cars there may be still room for improvement.

15. Try to find the best production plan where the profit is maximum and the amount of steel left over is minimum. Support your plan with data.

Section 12-2: The Mathematical Model (Appendix C, Section 2)

2.1 General discussion of a mathematical model and translation process.

2.2 Develop the "Profit" equation (Appendix C, Section 2), and graph different positions of this line segment in the first quadrant.

2.3 Develop the "steel restriction" inequality and graph the region related to it. Be sure to include in the discussion the inequalities $C \geq 0$, $T \geq 0$.

2.4 Develop the "plant capacity" inequality and graph the region related to it.

2.5 Discuss the solution of the system

$$\begin{aligned} C \geq 0 \ , \ T &\geq 0 \\ 1.5C + 3T &\leq 975{,}000 \\ C + T &\leq 500{,}000 \end{aligned}$$

for which

$$P = 300C + 400T \text{ is a maximum.}$$

2.6 Emphasize need for studying "Systems of Sentences" apart from the problem in order to develop efficient machinery for analyses of such decision problems.

Section 12-3: Solution sets of systems of equations and inequalities

1. Review definition of solution set of an equation or inequality.
2. Define solution set for systems of equations and inequalities. Give examples in which the solution set contains no ordered pairs, 1 ordered pair, infinitely many; e.g., for equations,

$$\begin{cases} 2x + y = 5 \ , \\ 2x + y = -2 \ . \end{cases} \qquad \begin{cases} 2x + y = 5 \ , \\ 2x - y = 5 \ . \end{cases} \qquad \begin{cases} 2x + y = 5 \ , \\ 6x + 3y = 15 \ . \end{cases}$$

At this point the solution sets may be found by examining the graphs!

Section 12-4: Equivalent equations and equivalent systems of equations

1. Two equations or two systems of equations are equivalent if they have the same solution sets.
2. If an equation in a system of equations is replaced by an equivalent equation the resulting system is equivalent to the original system.
3. Linear combination of left members of two equations (when right member is zero) used to construct simpler equivalent system of equations. (See Int. Math., Ch. 7, pp. 374-81; also FCA, Ch. 15, pp. 468-484.)

   Example:
   $$\begin{cases} 2x - y - 5 = 0 , \\ x - 3y + 5 = 0 . \end{cases}$$

   First replace $2x - y - 5 = 0$ by $a(2x - y - 5) + (x - 3y + 5) = 0$ for an appropriate a. An appropriate one is $a = -3$. The resulting equivalent system is
   $$\begin{cases} -5x + 20 = 0 , \\ x - 3y + 5 = 0 . \end{cases}$$

   Now replace $x - 3y + 5 = 0$ by $a(-5x + 20) + b(x - 3y + 5) = 0$. One might choose $a = \frac{1}{5}$ and $b = 1$ or one might take $a = 1$ and $b = 5$.

   We eventually get the equivalent system:
   $$\begin{cases} x = 4 , \\ y = 3 ; \end{cases}$$
   for which the solution set is clearly $\{(4,3)\}$.

Section 12-5: Systems of Linear Equations

1. Review of graphic solution.
2. Graphic interpretation of linear combination and solution sets.

   Family of lines through a point.

   A look at the possible cases:

   L_1 and L_2 the same line
   $$\begin{cases} x + y = 2 , \\ 2x + 2y = 4 . \end{cases}$$

L_1 and L_2 parallel:

$$\begin{cases} x + y = 2 , \\ 2x + 2y = -5 . \end{cases}$$

L_1 and L_2 intersect in a single point:

$$\begin{cases} x + y = 2 , \\ x - y = 4 . \end{cases}$$

Section 12-6: Graphic Solution of Systems of Inequalities

(See FCA, pp. 485-492.)

Many examples of increasing difficulty and complexity; e.g.,

1. $\begin{cases} y < x , \\ x > 2 . \end{cases}$

2. $\begin{cases} |y| < 2 , \\ |x| < 1 . \end{cases}$

3. $\begin{cases} 2x + 3y < 1 , \\ x - y > 2 . \end{cases}$

4. $\begin{cases} 2x + 3y < 2 , \\ 2x + 3y > 0 . \end{cases}$

5. $\begin{cases} 2x + y > 2 , \\ x + y < 1 , \\ x \leq 3 . \end{cases}$

6. $\begin{cases} x + y \leq 5 , \\ y \leq 3x + 4 , \\ y \leq -x + 4 . \end{cases}$

7. In the intersection of three half-planes which will define a triangular region emphasize geometric part.
8. Distinguish unbounded and bounded situations.

Section 12-7: Applications

1. State some problem situations which really need two variables to represent the conditions. Give the students an opportunity to formulate systems of equations and use the methods developed earlier in the chapter.
2. Discussion of "The Mathematical Model." (Appendix C, Section 2) Try to create a class situation so that the students will develop for themselves some of the characteristics of the process of "modeling".

We are now going to discuss the translation of a stated problem into a mathematical one. First, we must replace the words we have used to describe the situation in ordinary language by mathematical symbols and relations. This will provide a mathematical model. It is only a model of the real situation because (1) we can never list and include all the facts, only those facts that we think are most important, and (2) we cannot know the exact relationships in the real situation, so our mathematical relations will only be approximations to the real life situation.

A great advantage of a mathematical model is that you can do "experiments" with it just with pencil and paper or computer. You can say, "What would happen if such and such were done?" Then, you can carry out the mathematics and find out what the model predicts. You don't have to build something in a laboratory and test it or wait until it happens in the real world. Often a mathematical model is the only way to get such information when no laboratory experiment is possible. For instance, when you want to determine the route to be travelled to the moon by the first manned spaceships.

If our model is complete enough it will provide a good enough approximation to the real life situation so that we can rely on the answers it gives us. Of course the best test we have is to compare the predictions made by the model with the real situation and see how well they agree. Eventually this must always be done. If the agreement is poor we may have to add more features to the model. You can see that many different models can be made for the same real life situation just as an artist can depict a scene in many different ways.

3. Develop some simple linear programming problems where the student can use the techniques of Section 12-6 to find a convex region over which we wish to maximize a certain function of two variables. (Don't get too complicated -- the ninth grade unit will develop this section.)
   (a) Use simple production, transportation, and diet problems to illustrate the usefulness of this idea in different contemporary business situations.
   (b) Concentrate on the geometric interpretation of the situations.

[For an exposition appropriate for Junior High and as a source of problems at this level, see Chapter 7, pp. 212-222 of Some Lessons in Mathematics, edited by T. J. Fletcher, Cambridge University Press, 1965.]

Grade 9 - Listing of Chapters

Grade 9 (1966)	Grade 9 (1967)
1. Exponents, Logarithms, Slide Rule	1. Exponents, Logarithms, Slide Rule
2. Transformations	2. Deductive Reasoning - Logic
3. Systems of Sentences	3. Systems of Sentences and Optimization
4. Systems of Sentences	4. Measure Functions and Their Properties
5. Measure Theory	5. Statistics
6. Statistics	6. Displacements - Vectors
7. Deductive Reasoning	7. Transformations
8. Vectors	8. Circular Functions
9. Circular Functions	9. Tangency
10. Tangency	10. Measure
11. Measure	11. Complex Numbers
12. Complex Numbers	

OUTLINES OF GRADE 9 CHAPTERS

Grade 9 - Chapter 1

Exponents, Logarithms, Slide Rule

1966 Outline, pp. 324-330.

Grade 9 - Chapter 2

Deductive Reasoning - Logic

The 1966 outline contains the article, "The Role of Logic in Elementary Mathematics," pp. 479-484.

The following outline replaces the 1966 outline, pp. 353-355.

Preamble:

It is recommended that the following chapter serve as a replacement for the original chapter on Deductive Reasoning for Grade 9.

It should be observed, at the beginning of this discussion, that we are not suggesting that a formal study of logic be introduced in the secondary schools. However, logic is fundamentally the grammar of mathematics; it provides a way of organizing mathematical ideas; and it provides a way of clarifying their meaning. In the learning of mathematics, the new ideas or techniques are justified and related to the total scheme through logic. This process largely determines the learning sequence of mathematical concepts, and it is important that the student should become aware of how the ideas of mathematics are connected and that they do not stand in isolation.

The mathematician has long since developed the thought patterns associated with logical reasoning and uses such patterns in almost an infinite variety of ways as he works. Novice students in mathematics are not naturally aware of these patterns of thought, and are not adept in the use of such reasoning patterns. We hope to identify those intuitive aspects of logic that occur in most mathematical arguments and emphasize them as they repeatedly occur in context.

We approach logic at this level as a study in validity. We recommend an operational treatment: "What does it take to know that a statement is true?"

Background:

1. Students have seen and been involved in a large number of logical arguments in geometry, algebra, and number theory. Some of these proofs have been quite informal while others have been highly structured.
2. Students have had the opportunity to observe, and sometimes even participate in, the process of extending mathematical concepts by sequences of logical arguments.
3. Students have not had the opportunity to develop any definite ideas about the structure or validity of such arguments.

Purpose:

1. To develop an understanding of how the meanings of mathematical statements are determined.
2. To develop an understanding of how mathematical statements are used in an argument.
3. To present an extraordinarily useful language for the formulation and/or comprehension of mathematics in its development as a logical sequence of ideas.
4. To develop a clear understanding of the notion of proof.
5. To develop an understanding of the role of the single connectives, "and", "or", and "not".
6. To develop an understanding of the conditional and biconditional statements in mathematics.
7. To develop a clear understanding of the role of Implication and Equivalence, the rules of inference.
8. To develop an understanding of the nature of a valid argument.

Section 1. A Problem. (Class Discussion)

1-1. In a certain unmentionable community, politicians always lie, and non-politicians always tell the truth. A stranger meets three citizens, and asks the first of them if he is a politician. The first citizen answers the question. The second citizen then reports that the first citizen denied being a politician. Then the third citizen asserts that the first citizen is really a politician.

1-2. Some questions: (Class Involvement)*

1. Do you think it is possible to arrive at any conclusions about the occupations of the citizens? If so, what?
2. What do you need to do to support your conclusion, if you could find one?
3. How could you be convinced that your conclusion was valid?
4. What seems to be the source of your difficulty, if any, in analyzing this confusing community situation?

*(To the writers, and for the T.C.: This problem, and its accompanying questions, is a "simple-minded" attempt to motivate the need for some "machinery" to do some logical thinking. Even more important, once a conclusion is reached, we want the student to begin to realize that this conclusion can be proved correct if he can construct a valid argument whose premises are contained in the problem, and whose conclusion is the answer to the problem. Incidentally the question I have in mind is, "How many of these three citizens are politicians?" Maybe students will see more interesting questions, and the teacher should be cautioned not to "shortstop" such variations at this time. It is hoped that none of the students will be able to formulate a conclusion to the above question and prove it, but the teacher might help add to the confusion by "urging" students to assume all three, or the first two, etc., are politicians, and to test these assumptions informally in the problem. In any case, leave the student with the feeling that he will eventually be able to clarify this situation and justify his conclusion.)

Section 2. Statements and Connectives.

2-1. Logic, like most branches of mathematics, begins as a set of ideas communicated through words, which when condensed to some symbolic form are capable of a systematic treatment of transformation, simplification and equivalence.

2-2. "n is a positive integer" is an example of a sentence.
"$n^2 - n + 41$ is a prime" is another example of a sentence.

STATEMENTS: Declarative sentences which are capable of being considered "True" or "False". We are not directly concerned as to how they are judged to be "True" or "False": frequently it is through a common agreement as to the meaning of words and the context of the statement; i.e., "Red is a color", or "The sum of two and three is five", or "Three is more than ten": or it may be based on a common background of experience; i.e., "The sun appears to rise in the East", or "The Los Angeles County Fair is held in September". The decisive quality is that it makes sense to say "This statement is true" or "This statement is false".

Statements are usually represented symbolically by letters near the middle of the alphabet -- p , q , r , s , etc.

2-3. Generalizations using "and" and "or".
"For each n , n is a positive integer AND $n^2 - n + 41$ is a prime."
Another alternative is,
"For each n , n is a positive integer OR $n^2 - n + 41$ is a prime."

(To the writers: I feel that at this point, at least indicate the possibility in the T.C., that it would be worthwhile to do some experimenting, hopefully leading to the discovery that the generalization, "For each n , $n^2 - n + 41$ is a prime", is false. Then, also hopefully, develop a classroom argument about whether the above two generalizations are true, when the generalization, "For each n , n is a positive integer" is true.)

CONNECTIVES:

AND: When two or more statements are compounded through the use of the connective "and", the result is a new statement called the CONJUNCTION of the original statements, and the connective is represented symbolically by "$\wedge$ ". "$p \wedge q$" may be read as "the conjunction of p and q" , or more simply as "p and q" .

OR: When two or more statements are compounded through the use of the connective "or", the result is a new statement called the DISJUNCTION of the original statements.

In common usage, the word "or" has two distinct meanings, depending on whether it is intended to include the possibility of "both". When we say "I am planning on going to the beach or to a show this weekend" we certainly intend to include the possibility of doing both, which conforms to the usage in legal documents of the connective "and/or". It is described as the "inclusive or" and symbolized by " $\vee$ " . (This is the meaning of "or" most used in mathematics.) On the other hand, when we say "The lights are on or off" we certainly exclude the possibility of "both" and are then using the "exclusive or", which is occasionally needed and is symbolized by " $\underline{\vee}$ " .

2-4. Negation.

NOT: Any statement combined with the phrase "It is not the case that", or simply "not", is called the NEGATION or DENIAL of the original statement, and the negation itself is symbolized by " $\sim$ " . Thus, if "q" represents the statement "The moon is made of green cheese", then "$\sim q$" represents the statement "It is not the case that the moon is made of green cheese", or more simply "The moon is not made of green cheese".

When two or more statements are compounded by the use of any of the above connectives (or some others which we will meet later) the original statements are called the "components" of the connective(s) in the compound.

2-5. Sample exercises.

(1) Form the denial of each of the following and then rephrase it if necessary in good idiomatic English. Your final answer should not use the phrase "It is not the case that" but should have this meaning:

(a) He saw me coming.

(b) The largest number whose square is less than 40 is 5 .

(c) Jones caught eight fish.

(d) Jones caught more than eight fish.

(e) Jones has a large family.
(f) Smith can outbid every competitor.
(g) Everyone is entitled to an opinion.
(h) Jones caught the fish and Brown caught a cold.
(i) All who succeed are virtuous.
(j) Fame is sometimes sweet.

(2) Phrase each of the following as conjunctions of the simplest possible components:

(a) The Governors of both New York and Illinois are Republicans.
(b) May and June are bright and gay.
(c) King Henry IV, deposer of his cousin Richard II, was father of Henry V.
(d) Jones and Smith deserve support.
(e) Jones and Smith are helping each other.

(3) Let p represent the statement "The night is young", and let q represent the statement "You are beautiful".

Give the verbal version of each of the following as simply as you can:

(a) $\sim q$
(b) $p \vee q$
(c) $\sim p \vee q$
(d) $p \wedge \sim q$
(e) $\sim p \wedge \sim q$
(f) $\sim \sim p$
(g) $\sim (p \wedge q)$
(h) $\sim p \vee \sim q$
(i) $\sim (p \vee q)$
(j) $\sim (p \vee \sim q)$

Logical Counterparts:

When any statement, simple or compound, is written in symbolic form we call this the "symbolic counterpart" of the original statement. Note that many statements may have the same symbolic counterpart if they have the same structure. In this case, each such statement is called an "instance" of the symbolic counterpart. Note also the parallel here between a "word-problem" and its model.

In each of the following, identify each simple statement by a letter and then write the symbolic counterpart of the complete statement

In terms of these letters:

Example 1: "John played golf, or Henry played tennis and Frank did not go swimming." Let: p = "John played golf", q = "Henry played tennis", r = "Frank went swimming".

Ans: $p \vee (q \wedge \sim r)$.

Example 2: "John played golf or Henry played tennis, and Frank went swimming." (Using same letters)

Ans: $(p \vee q) \wedge r$.

(Note that in each case, the symbolic grouping reflects the punctuation of the original compound statement as it would in algebra. Note also that the letters are normally chosen to represent positive statements, using the denial where necessary.)

(4) a is a multiple of both 3 and 5 .

(5) b is divisible by neither 3 nor 5 . (careful!)

(6) c is an even number larger than 29 .

(7) c is an even number larger than 29 and less than 35 .

Section 3. Logical Values.

Our logic is two valued, which means we assume that a statement is either true or false. We assume that there is a truth value function which assigns to each statement the value "1" (truth) or "0" (falsity). Now, since the logical value of a compound must depend solely on the logical value of its components combined with the logical pattern associated with its connectives, it should be possible to compute the logical value of any compound statement, no matter how complex, entirely from these parts.

The following tables show the effect of each type of connective on the logical values of its components. In each case we are assuming p and q are statements.

Negation:

$\sim p$	1	0
	0	1

or, in word form: "The DENIAL of a statement is a statement having exactly the opposite logical value of its component."

Conjunction:

		q	
	p ∧ q	1	0
p	1	1	0
	0	0	0

or: "The CONJUNCTION of two statements is a statement having the logical value 'TRUE' just in case the logical values of both components are 'TRUE', and 'FALSE' otherwise."

Disjunction:

		q	
	p ∨ q	1	0
p	1	1	1
	0	1	0

or: "The (inclusive) DISJUNCTION of two statements is a statement having the logical value of "FALSE" just in case the logical values of both components is "FALSE", and "TRUE" otherwise."

(We will have little or no use for the exclusive disjunction in mathematics.)

3-1. Sample exercises.

(1) Determine the logical value of each of the following compound statements, where all numerals represent Natural Numbers:

(a) $2 + 2 = 4$ and $3 + 4 = 12$.
(b) $2 \cdot 2 = 4$ or $3 \cdot 3 = 6$.
(c) $2 + 2 = 4$ or it is not the case that $3 \cdot 3 = 6$.
(d) $2 \cdot 2 = 4$ and $3 \cdot 3 \neq 6$.
(e) $2 + 2 = 4$ and $3 \cdot 4 \neq 12$.
(f) $3 + 4 = 12$ or $4 + 3 = 12$.

(2) Construct compound statements as required (e.g., one true answer to (a) is p ∨ q .)

(a) From p = "$2 + 2 = 4$" and q = "$3 + 4 = 12$" .
Two (other) true statements: __________ __________
Three false statements: __________ __________ __________

(b) From r = "$2 + 2 = 4$" and s = "$3 \cdot 4 = 12$" .
Two true statements: __________ __________
Two false statements: __________ __________

(3) (Some help needed here for the student in how to do these exercises.) Make up your own program and compute the logical values of each of the following:

(a) $\sim(p \wedge q)$

(b) $\sim p \vee \sim q$

(c) $\sim p \wedge \sim q$

(d) $\sim(p \vee q)$

(e) $(p \wedge q) \vee \sim r$

(f) $p \vee (\sim q \wedge r)$

(g) $\sim(p \wedge \sim q)$

(h) $\sim[p \wedge (\sim p \vee q)] \vee q$

(i) $(\sim p \vee q) \wedge (p \vee \sim q)$

(j) $(\sim p \vee \sim q) \wedge (p \vee q)$

(You will note that some pairs of the above come out with exactly the same ordered sets of logical values. We will make use of this later.)

Section 4. Conditionals and Biconditionals.

Having assimilated the basic connectives "and", "or", and "not", which are so common in everyday thought (just stop and think how many times a day you use these words!) we come to one which is more important mathematically but which, unfortunately, is less well understood.

A great many mathematical statements take the form (expressed or implied) of "If - - , then - - " or one of its variations, which indicates a relationship between a "cause" and its "effect"; a "hypothesis" and its "conclusion"; an "initial fact" and its "dependent fact". Statements of this kind lie at the heart of all proof, not only in geometry but throughout mathematics.

In all of its usages there is a sense of "flow" from one fact to another, so it is natural that it be symbolized by "$\Rightarrow$" , and "if p , then q" becomes "$p \Rightarrow q$" . The compound formed by any two statements and "$\Rightarrow$" is called a CONDITIONAL and may be read descriptively as "p arrow q" , or its meaning may be indicated by "if p , then q" , "p only if q" , "q if p" , "q provided that p" , "q in case p" , "p is a sufficient condition for q" , "q is a necessary condition for p" . Many of these are inconvenient as not being left-to-right readings of the symbols and others seem to be obscure as to their meaning. Probably we had better use "p arrow q" until we are sure of its actual meaning and can use the other forms intelligently.

In determining the effect of the Conditional, it is obvious from its usage that "$p \Rightarrow q$" must have a logical value of "True" (or 1) if p is "True" and q is "True" and a logical value of "False" (or 0) if p is "True" and q is "False". But what if p is "False"? What should we say of a statement such as "If two angles are right angles, then they are equal" if the problem on which we are working is not concerned with right angles? Our first reaction would probably be to say that the statement is neither "True" nor "False" under these conditions, or at least that it doesn't matter which, but these merely evade the question.

The position to be taken here is that if p is "False" (0), then it is possible to infer either a "True" or a "False" statement. Hence we will say that the Conditional must be considered "True" whenever the hypothesis is "False" whether the conclusion is "True" or not, and the complete table for the Conditional must be:

		q	
	$p \Rightarrow q$	1	0
p	1	1	0
	0	1	1

or in words: "The CONDITIONAL has a logical value of 'False' just in case the hypothesis is 'True' and the conclusion 'False', and of 'True' otherwise.

With these logical values in mind, the alternate readings of "$p \Rightarrow q$" given above should now take on meaning. Certain variations are also suggested:

"p if q" as "$p \Leftarrow q$" , a Reversed Conditional
"$q \Rightarrow p$" , called the CONVERSE of "$p \Rightarrow q$"
"$\sim p \Rightarrow \sim q$" , called the INVERSE of "$p \rightarrow q$"
"$\sim q \Rightarrow \sim p$" , called the CONTRAPOSITIVE of "$p \Rightarrow q$"

and the BICONDITIONAL, symbolized as "$p \Leftrightarrow q$" and meaning "$(p \Rightarrow q) \wedge (p \Leftarrow q)$" . The Biconditional is met so frequently in geometry in the form "If - -, then - -, and conversely", and in algebra in the form " - - if and only if - - " and plays such an important role logically in establishing equivalence that it deserves a complete description as:

		q	
	$p \Leftrightarrow q$	1	0
p	1	1	0
	0	0	1

or in words: "The BICONDITIONAL has the logical value "True" just in case the logical values of the components are exactly the same, and of "False" otherwise.

For the Teacher's Commentary:

An alternate "justification" for the standard truth table "$A \Rightarrow B$" could be done as follows:

(1) When A and B are both true, then common sense suggests $A \Rightarrow B$ is true.

(2) When A is true while B is false then common sense suggests $A \Rightarrow B$ is false.

Now most students will agree that "$A \Rightarrow A$" (i.e., (If A then A) should always be true; regardless of the truth of A since we say "If A ..."). However, for $A \Rightarrow A$ to always have the value True, we require that when A is false then $A \Rightarrow A$ is true. Hence it seems plausible that

(3) When A is false and B is false, then $A \Rightarrow B$ is true.

Finally, most students will agree that whenever

"If C then D" and "If D then E"

hold, then

"If C then E"

holds. It would seem that transitivity is a basic tool of logic and that it should hold regardless of the truth values of C , D , and E . Now therefore this will hold even if $C \Rightarrow D$ is false; say when C is True while D is False. Thus when E is True we have

$$\underbrace{\underbrace{\underset{\text{False}}{[(C \Rightarrow D)} \quad \text{and} \quad \underset{\text{(immaterial)}}{(D \Rightarrow E)]}}_{\text{False}} \Rightarrow \underbrace{(\underset{\text{True}}{C} \Rightarrow \underset{\text{True}}{E})}_{\text{True}}}_{\text{True}}$$

In other words, in general

(4) When A is false while B is true, then $A \Rightarrow B$ is true.

4-1. Sample exercises. (Some Geom. examples needed)

(1) Determine the logical value of each of the following composite statements. (All numerals represent Real Numbers):
   - (a) If $2 + 2 = 4$, then $3 + 3 = 0$.
   - (b) If $2 + 2 = 4$, then $3 + 3 = 6$.
   - (c) If $2 + 2 = 5$, then $3 + 3 = 7$.
   - (d) If $2 + 2 = 5$ or $3 + 3 = 6$, then $5 + 5 = 11$.
   - (e) If $2 + 2 = 5$ and $3 + 3 = 6$, then $5 + 5 \neq 11$.

(2) Calculate the complete table of logical values for each of the following:

(a) $(p \Rightarrow q) \Rightarrow \sim(q \Rightarrow p)$.

(b) $(p \Rightarrow q) \wedge (q \Rightarrow r)$.

(c) $\sim(p \wedge q) \Leftrightarrow (\sim q \vee \sim p)$.

(d) $(p \Leftrightarrow \sim q) \Rightarrow (\sim p \wedge q)$.

(e) $(p \Leftrightarrow q) \Leftrightarrow [(p \wedge q) \vee (\sim p \wedge \sim q)]$.

(3) Make complete evaluations of the patterns below. Describe how the two tables for each pair are related:

(a) 1. $p \Rightarrow (q \Rightarrow r)$
    2. $(p \wedge q) \Rightarrow r$
(b) 1. $p \Rightarrow q$
    2. $\sim q \Rightarrow \sim p$
(c) 1. $\sim p \Rightarrow q$
    2. $p \vee q$
(d) 1. $\sim(p \wedge q)$
    2. $\sim p \vee \sim q$
(e) 1. $\sim(p \Rightarrow q)$
    2. $p \wedge \sim q$
(f) 1. $p \Leftarrow q$
    2. $\sim p \Rightarrow \sim q$

Section 5. Basic Logical Equivalences.

(This section should be written so that the student and the teacher realize that it is intended to be largely informational. There should be no attempt to "drill and memorize". At best it should be considered as a reference section, and as a section which lightly tries to tie some things together and point out some possible future activities. I do think it would be appropriate to illustrate with examples the "miscellaneous laws" of proof, drawing upon the students' background in geometry, number theory, and algebra.)

Below are listed some of the more basic logical equivalences together with their descriptive titles. In these p , q , r represent statements with variable logical value as we have been using them, while T represents a statement with a constant value of "True" and F the corresponding "False" constant:

Identity Laws:

$p \wedge T \Leftrightarrow p$ $p \vee T \Leftrightarrow T$ $(p \Leftrightarrow T) \Leftrightarrow T$
$p \wedge F \Leftrightarrow F$ $p \vee F \Leftrightarrow p$ $(p \Leftrightarrow F) \Leftrightarrow \sim p$

Complementary Laws:

$p \wedge \sim p \Leftrightarrow F$ $p \vee \sim p \Leftrightarrow T$ $(p \Leftrightarrow \sim p) \Leftrightarrow \sim p$

Idempotent Laws:

$p \wedge p \Leftrightarrow p$ $p \vee p \Leftrightarrow p$ $(p \Leftrightarrow p) \Leftrightarrow T$

DeMorgan's Laws (Denial):

$$\sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$$

$$\sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q$$

$$\sim(p \Rightarrow q) \Leftrightarrow p \wedge \sim q$$

$$\sim(p \Leftrightarrow q) \Leftrightarrow (\sim p \Leftrightarrow q)$$

Commutative Laws:

$$p \wedge q \Leftrightarrow q \wedge p$$

$$p \vee q \Leftrightarrow q \vee p$$

$$(p \Rightarrow q) \Leftarrow\!X\!\Rightarrow (q \Rightarrow p)$$

$$(p \Leftrightarrow q) \Leftrightarrow (q \Leftrightarrow p)$$

Associative Laws:

$$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$$

$$p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$$

$$p \Rightarrow (q \Rightarrow r) \Leftarrow\!X\!\Rightarrow (p \Rightarrow q) \Rightarrow r$$

$$p \Leftrightarrow (q \Leftrightarrow r) \Leftrightarrow (p \Leftrightarrow q) \Leftrightarrow r$$

Distributive Laws:

$$p \wedge (q \vee r) \Leftrightarrow [(p \wedge q) \vee (p \wedge r)]$$

$$p \vee (q \wedge r) \Leftrightarrow [(p \vee q) \wedge (p \vee r)]$$

$$[(p \vee q) \Rightarrow r] \Leftrightarrow [(p \Rightarrow q) \wedge (q \Rightarrow r)]$$

$$[(p \wedge q) \Rightarrow r] \Leftrightarrow [p \Rightarrow (q \Rightarrow r)]$$

Miscellaneous Laws:

$[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow (p \Rightarrow r)$ [Transitive Law of the Conditional]

$(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$ [Indirect proof of a Conditional](contrapositive)

$\sim(p \Leftrightarrow q) \Leftrightarrow (\sim p \Leftrightarrow q) \Leftrightarrow (p \Leftrightarrow \sim q)$ [Disproof of a Biconditional]

$(p \Leftrightarrow q) \Leftrightarrow (\sim p \Leftrightarrow \sim q)$ [Indirect proof of a Biconditional]

If you were at all observant as you read through this list, it must have occurred to you that almost all of the properties of the Real Number System (except "order") are present here and hence that there must exist an Algebra of Logic which very closely parallels the Algebra of the Real Numbers. We are very close to an Arithmetic of Logic, which is in fact exactly the basis of electronic "decision making" circuits! We have no intention of taking time for this now, but many of you will probably have an opportunity to develop both the Arithmetic and the Algebra in some of your future courses.

There is another Algebra which is so closely connected to the Algebra of Logic and so easily derived from it that we cannot afford to ignore it at this time. We have found the ideas and notation of Sets very useful and have no doubt found many instances in which an answer might correctly be written in several different forms (an "equivalence"), and thus have suspected that an Algebra of Sets exists and could be put to good use. The Algebra of Sets is sometimes called "Boolean Algebra" after George Boole (1815-1864) and is essential to a complete understanding of the use of Sets.

In the "set-builder" notation we define a set in terms of some logical statement -- those items for which the statement has a logical value of "True" become elements of the set; those items for which the statement has a logical value of "False" are not elements of the set and hence become elements of the "complement" of the set. For instance: $A = \{x : 2x + 3 > 5\}$ defines the solution set of the inequality $2x + 3 > 5$ within the domain of the inequality. In other words: any value of x within the domain which makes the statement "$2x + 3 > 5$" have a logical value of "True" becomes a member of the set $(x \in A)$ and any value of x which makes the statement have a logical value of "False" lies outside the set $(x \in A')$. The Union and Intersection of sets are logically defined as:

$$A \cup B = \{x : (x \in A) \vee (x \in B)\} \qquad A \cap B = \{x : (x \in A) \wedge (x \in B)\}$$

With $U = \{x : x$ lies within the domain of our statement$\}$, our constant T in the preceding list becomes U; our constant F becomes $\emptyset$; $\vee$ becomes $\cup$; and $\wedge$ becomes $\cap$. There is no "set" counterpart for "$\Rightarrow$" or "$\Longleftrightarrow$", but we already have the full list of basic Set Equivalences which forms the basis of Boolean Algebra:

Identity:	$A \cap U = A$	$A \cup U = U$
	$A \cap \emptyset = \emptyset$	$A \cup 0 = A$
Complements:	$A \cap A' = \emptyset$	$A \cup A' = U$
Idempotent:	$A \cap A = A$	$A \cup A = A$
DeMorgan:	$(A \cap B)' = A' \cup B'$	$(A \cup B)' = A' \cap B'$
Commutative:	$A \cap B = B \cap A$	$A \cup B = B \cup A$
Associative:	$A \cap (B \cap C) = (A \cap B) \cap C$	$A \cup (B \cup C) = (A \cup B) \cup C$
Distributive:	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

It is the strong similarities between the "structures" of Numerical Algebra, Logic Algebra, and Boolean Algebra which make our study of "structure" so vital and it is hard to say which you will find more useful in your mathematical future. Fortunately, we need not worry about it because once you have any one of them properly understood, you have them all!

This is one reason that "structure" has become so important in the study of mathematics at all levels from kindergarten through college and into practical application, especially in an Age of Automation.

Section 6. A Rule of Inference.

Thus far we have studied only how we assign the truth values to statements when we know the truth values of their parts. This does not tell us how we prove things. That is, how we can infer that one statement, B , has the truth value 1 from other statements. There is one major rule for doing this. Suppose that we know that the conditional

(1) $A \Rightarrow B$ has the truth value 1 ,

and suppose we know that

(2) A has the truth value 1 ;

then it follows from the truth table for the conditional that

(3) B has the truth value 1 .

In this way we could establish that the truth value of B is 1 without knowing ahead of time that this was so. Note that we must determine both the truth value of $A \Rightarrow B$ and A to use this rule. It may seem a bit of a paradox that we can determine the truth value of $A \Rightarrow B$ without first knowing the truth value of B . It is not a paradox because part of the truth table for $A \Rightarrow B$ gives the value 1 regardless of the truth value of B ; namely when A has the truth value 0 . Thus we have only to deal with the cases when A has the truth value 1 ; that is, we can assume that A is true. Naturally, this makes it easier. But to get new information about B we then must also establish that A has the truth value 1 .

Example 1: Consider the sentence

If 3 divides 5,043 then 3 divides 10,086 .

Here, of course

A is "3 divides 5043" and

B is "3 divides 10086" .

Now the truth of $A \Rightarrow B$ is established by noting that $10086 = 2 \cdot 5043$ and so if 3 does indeed divide 5043 , say $5043 = 3 \cdot n$, then it follows that $10086 = 2 \cdot 3 \cdot n$ or that 3 divides 10086 . Thus when A has the truth value 1 it follows that B has the truth value 1 . Hence the truth value of $A \Rightarrow B$ is 1 . But now to infer that B has the truth value 1 (that is, that 3 does divide 10086) we must check to see that 3 divides 5043 . Upon division we find that $5043 = 3 \cdot 1681$ and so we know that 3 does divide 10086 .

Example 2: If 3 divides 5086 then 3 divides 10172 .

We shall not repeat the whole of the argument above. It should be clear that the demonstration we have given to show that $A \Rightarrow B$ has the truth value 1 goes over here by just replacing 5043 by 5086 and 10086 by 10172 . However, we cannot demonstrate that 3 divides 5086 and so we cannot infer that 3 divides 10172 . As far as this argument shows, it may or it may not. To decide which we shall have to do more mathematics.

Section 7. Quantification.

7-1. "For all" and "There exists".

When we have a statement $A(x)$ which depends upon a variable x then "For all x , $A(x)$" is assigned the truth value 1 if indeed $A(t)$ is true for all possible t in the range of A . Otherwise it is assigned the value 0 . Similarly "There exists an x such that $A(x)$" is assigned the truth value 1 if for at least one t in the range of A , $A(t)$ is true; otherwise the sentence is assigned the value 0 .

The notion of "range" as used here can be left rather vague and tenuous -- just as we have left the notion of variable.

7-2. Negation: The negation of (For all x , $A(x)$) is trivially,

$$\sim(\text{For all } x , A(x)) ,$$

which is equivalent to

$$\text{There exists } x \text{ such that } \sim(A(x)) .$$

The negation of (There exists x such that $A(x)$) is equivalent to

$$\text{For all } x , \sim(A(x)) .$$

The assignment of truth values establishes these assertions. Formal verification is probably not as convincing as demonstrations with specific examples.

7-3. (Sample exercises needed to amplify this notion.)

Section 8. Other Rules of Inference.

There are two other rules of inference which are important.

8-1. If "For all x , $A(x)$" has the truth value 1 then we infer that $A(t)$ has the value 1 for each t in the domain. Conversely, if $A(t)$ is true, independent of t (that is, if t plays no role in the argument) then we infer "For all x , $A(x)$" .

8-2. If $A(t)$ has the truth value 1 (for some specific t in the domain of A) then we infer "There exists x such that $A(x)$" .

Section 9. Axioms and Theorems.

An axiom is a specific statement to which we arbitrarily assign the truth value 1 .

Example: For all pairs of integers, $a + b = b + a$.

A theorem is a statement whose truth value we have determined to be 1 . (Thus we regard axioms as theorems.)

Section 10. Proof.

A proof of a statement is a logical argument we make to demonstrate that the statement is a theorem, i.e., that it has the truth value 1 . Usually the statement in question has the form $A \Rightarrow B$ and it will usually have quantifiers as a part of the statement.

The strict logical definition of a proof of B from hypothesis $A_1 , \dots , A_n$ (i.e., a proof of $(A_1$ and A_2 and ... and $A_n) \Rightarrow B$) requires a sequence of sentences each of which is either an axiom of a previously proved result, or it follows from one of the rules of inference applied to statements which appear earlier in the sequence. This notion is closer to what should occur in the classroom since it is a reduction of a complicated sentence to a collection of simpler and more obvious statements.

Example: For all integers n , if n is odd then n^2 is odd.
(Form : For all n , $(A(n) \Rightarrow B(n))$.

Proof: For all integers n we must show that the truth value of $(n \text{ is odd}) \Rightarrow (n^2 \text{ is odd})$ has the truth value 1 . It will then follow (Section 9) that the statement is a theorem. To show $(n \text{ is odd}) \Rightarrow (n^2 \text{ is odd})$ we assume "n is odd" is true. Now:

$* \; n \text{ is odd} \Longleftrightarrow \text{there exists an integer, } k ,$

such that $n = 2k + 1$. (To shorten our example we shall suppose that this is a previously proved theorem; we should of course verify that this regression will occur until the axioms on which the integers are based are employed.)

If $n = 2k + 1$ then $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. This holds, as you will recall, because of the distributive laws. Thus $n^2 = 2(2k^2 + 2k) + 1$ and by (*) it follows that n^2 is odd . Hence $(n \text{ odd}) \Rightarrow (n^2 \text{ odd})$ has the truth value 1 .

For the T.C.: Here are a couple of philosophical remarks on proof which we offer:

1. A typical attitude toward proof, especially among practicing mathematicians is that "A proof is an argument that convinces the listener." This is a very practical view for those who are sufficiently concerned to doubt or care. As a philosophy it is totally unsuited to the high school classroom, since students seldom really care and certainly never doubt the authority of text or teacher.

2. An important part of a proof is to discover why a particular theorem holds. In the proof we find exactly what "makes it tick". For example, in proving $(n \text{ odd}) \Rightarrow (n^2 \text{ odd})$ how much depends upon "odd"? Would (essentially) the same proof hold for $(n \text{ even}) \Rightarrow (n^2 \text{ even})$ or for
$(n \text{ has the form } 3k + 1) \Rightarrow (n^2 \text{ has the form } 3k + 1)$ or for
$(n \text{ has the form } 3k - 1) \Rightarrow (n^2 \text{ has the form } 3k - 1)$?

3. Students need training in putting facts together to create new mathematics. In this game the alteration of hypotheses plays a key role. Was each hypothesis used? Can one hypothesis be dropped? If an hypothesis is added or dropped or denied, what is the change in the conclusion?

Finally we point out that this concept of proof has little to do with a strategy for proof. In trying to construct a proof we often make false starts, collect bits and pieces of evidence bearing on the proof, or change the form of the theorem to an equivalent form. In short, since we don't know at the start just what the sequence of steps shall be, we may have to do quite a lot of trial and error work until a suitable sequence arises. But the ultimate test of a proof is whether this sequence can be found.

Incidentally, mathematicians are satisfied (convinced) when it "becomes clear" that a proof sequence can be established. Seldom is a complete proof sequence written down.

Grade 9 - Chapter 3

Systems of Sentences and Optimization

Background:

1. Convex sets and intersections of convex sets.
2. Solution sets of systems of equations and inequalities, and their graphs.
3. Linear functions and their graphs.
4. An intuitive development of the linear programming problem.
5. Quadratic functions $f: x \rightarrow a(x-h)^2 + k$, equation of circles and parabolas.
6. Three dimensional coordinate system and algebraic description of subsets of space.

Purpose:

1. To extend the development of linear programming and its applications.
2. To refine the mathematical ideas involved such as convexity, polygonal convex set, and the extreme points of such a set.
3. To focus explicitly on the optimization problem of determining the maximum or minimum of a linear function defined over a polygonal convex set.
4. To review and extend the Grade 8 development of systems of linear equations.

5. To introduce and treat very lightly systems of first degree equations in three variables, and systems of one linear and one quadratic equation.

Section 3-1. Introduction.

(1) Pose a linear programming problem in two variables which is slightly more complicated and more succinctly stated than the one posed in Grade 8.

(2) Guide the formulation of the equation and inequalities from the statement of the problem.

(3) Develop a graphical solution to the problem in the manner found in Grade 8.

(4) Examine the solution process and highlight the aspects of the process which will be studied further in this section, namely,

   (a) the language of constraints in the form of inequalities,

   (b) the formation of polygonal convex sets by systems of sentences and noting extreme points,

   (c) the relationship between the linear function and the constraints provided by the inequalities,

   (d) the solution being found at an extreme point.

Section 3-2. Constraints and Inequalities.

(1) Note that decisions are made within certain boundary conditions or constraints. Give some everyday examples: Buying a dress or suit, etc.

(2) Provide some conditions which are to be translated into inequalities.

(3) Review graphing of inequalities. Develop terminology of half-plane and closed half-plane for these solution sets. Review convexity of half-planes.

(4) Discuss conjunctions of constraints and accompanying intersections of solution sets of inequalities. Discuss convexity of intersections of convex sets.

Section 3-3. Polygonal Convex Sets (Constraint Sets).

(1) Have students graph solution sets of systems of inequalities that lead to polygonal convex sets.

(2) Analyze these polygonal convex sets as

   (a) being the intersection of a finite number of closed half-planes (spaces). Note that each of these closed half-planes contains the set, and this leads to another definition of convexity of polygons.

   (b) being bounded or unbounded. A polygonal convex set is unbounded if it contains a ray.

(3) Do the reverse. Provide drawings of polygonal convex sets and have students provide the systems of inequalities which have them as solution sets.

(4) Provide problems such as follows (from Kemeny et al) minimum nutritional requirements:

	Phosphorus	Calcium
Adults	.02	.01
Child	.03	.03
Infant	.01	.02

Plot the convex set and state whether or not the following assertions are true.

   (a) If a child's needs are satisfied, so are an adults.

   (b) Both an adult's and an infant's needs are satisfied only if a child's needs are.

   Etc.

This ties in with previous chapter on logic.

(5) Provide a system of inequalities which contains a superfluous condition and have students find it.

Section 3-4. Extreme Points.

(1) Exhibit some extreme points of polygonal convex sets. Then have students identify them. (Pointing exercises)

(2) Give systems of inequalities and have students find coordinates of extreme points.

(3) Give the coordinates of the extreme points of a bounded polygonal convex set and have students provide the system of inequalities.

(4) Give some problems involving constraints and have students discuss the extreme points of the problem.

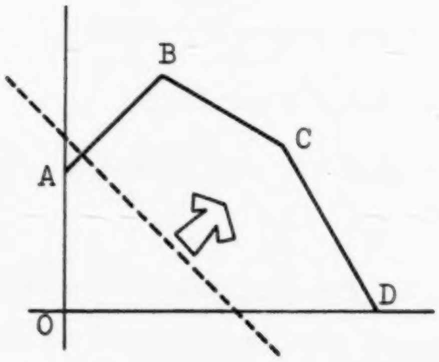
Section 3-5. Optimization of Linear Functions.

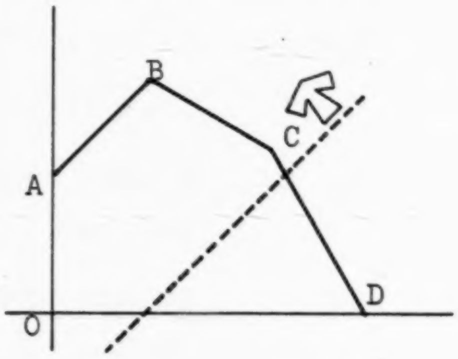
(1) Refer to problem posed in Section 1 and identify the linear function $f : (x,y) \to ax + by$ and the linear inequalities in x and y forming the polygonal convex set C .

(2) Discuss the intersection of the solution set for $ax + by = P$ and set C .

Discuss the effect of varying P and the solution occurring at an extreme point of C .

(3) Analyze similarly some sample situations.





At what point does the solution occur?

(4) Lead to the following statement:

> A linear function defined over a polygonal convex set C takes on its maximum and minimum value at an extreme point of C .

An informal proof of this is given in Kemeny et al.

(5) Develop the general method of finding the maximum or minimum of a linear function defined over a polygonal convex set C (bounded).

(a) Find the extreme points of C (there will be a finite number of them).

(b) Substitute coordinates of each into the function.

(c) The largest of the values will be the maximum and the smallest will be the minimum.

Section 3-6. Applications. (Appendix C, Section 3)

(1) Provide some linear programming problems.

Include both minimum and maximum problems.

Include cases where no solution exists,
where an infinite number of bounded solutions exist,
and at least one occurs at an extreme point
where the solution exists but is unbounded.

(2) Have students analyze how these cases arise.

Section 3-7. Systems of Linear Equations Revisited.

(1) Pose some problems leading to two linear equations in two variables.

Example: Consider the centigrade-Fahrenheit formula

$C = \frac{5}{9}(F-32)$ and pose these questions:

At what temperature do the two thermometers have the same reading?

At what temperature does the Fahrenheit thermometer have a reading 3 times that of the centigrade thermometer?

(2) Review graphical and algebraic procedures for finding solution.

Discuss equivalent equations and equivalent systems.

Review principle of linear combination.

(3) Consider cases of inconsistent and dependent systems and have students develop a graphical and algebraic analysis of these cases.

Examine these cases in terms of applications.

Section 3-8. Systems of First Degree Equations in Three Variables.

(1) Pose a problem leading to 3 first degree equations in 3 variables.

Example: (Modify diet problem found in Dorn-Greenberg text.)

(2) Develop a graphical analysis, showing that each is an equation for a plane and problem is finding point of intersection of three planes.

Show graphically how other cases may arise.

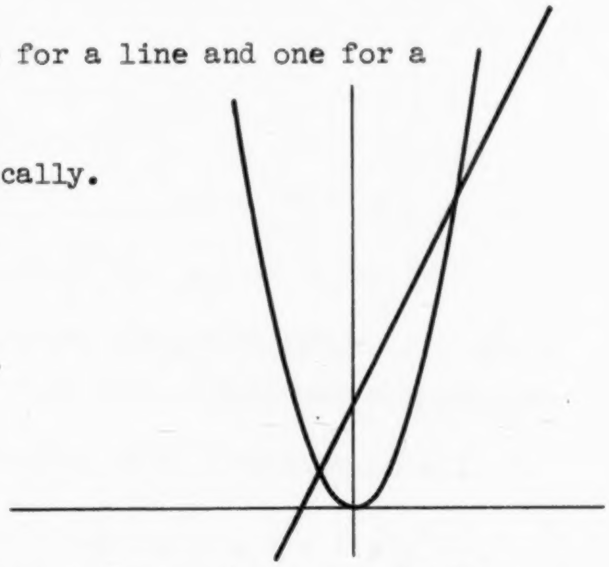
(3) Develop algebraic process for finding solution as an extension of process for two linear equations in the previous section.

(4) Since the process is repetitive, consider the possibility of developing a flow chart for the process, following Gauss's method. (See Dorn-Greenberg.)

Section 3-9. Systems of One Linear and One Quadratic Equation or Inequality. (Very lightly.)

(1) Consider pairs of equations, one for a line and one for a parabola, very simple cases.

Develop algebraically and graphically.



Example:
$$y = x^2$$
$$y = 2x + 3$$

Consider also
$$y \geq x^2$$
$$y \leq 2x + 3$$

and consider intersections of the family $y = 2x + k$.

(2) Carry out the same simple development for a circle and a line.

Example:
$$x^2 + y^2 = 5$$
$$x + 2y = 5$$

It is recommended that the following material not be included in the 7-9 sequence.

Systems of Sentences in Two or More Variables

Background:

1. In Grade 8 students will have studied systems of first degree sentences in two variables with a slight introduction to linear programming. (Chapter 12, 1967 sequence)

2. In Grade 8 students will have studied the quadratic function $f : x \rightarrow a(x - h)^2 + k$ extensively and be familiar with the equation of a circle. (Grade 7, Chapter 2, 1967 sequence)

3. They will not have any experience with equations of the hyperbola or ellipse, and will not be familiar with the general second degree sentence in two variables.

4. In Grade 9, Chapter 3, students will develop, hopefully, a higher level of sophistication in working with first degree sentences in two variables as they develop the linear programming chapter.

I recommend that a study of systems of sentences like:

$$\begin{cases} x + 2y - 3 = 0 \\ x^2 - 3x + 5 = y \end{cases}, \qquad \begin{cases} 2x^2 - 3x + 4 + y \leq 0 \\ x + 2y - 5 = 0 \end{cases},$$

$$\begin{cases} x^2 + y^2 - 3 = 0 \\ 2x^2 - 3x + 4 - 5y^2 = 0 \end{cases}, \qquad \vdots$$

along with their graphical representation be studied in conjunction with the appropriate sections in the 10 - 12 sequence.

I also recommend that systems of sentences like:

$$\begin{cases} x + y + z - 3 = 0 \\ 2x - 5y + 7z + 1 = 0 \\ 5x - 2y - 3z + 10 = 0 \end{cases}, \qquad \begin{cases} x + y + z + w - 4 = 0 \\ 2x - 5y + z - 3w + 10 = 0 \\ 5x + 10y - 7z + w - 2 = 0 \\ x + y - z - w + 2 = 0 \end{cases}, \quad \ldots$$

be included at appropriate places in the 10 - 12 sequence.

Some reasons for the recommendations:

(1) Students will have matrices to handle systems of first degree equations in two or more variables in the 10 - 12 sequence.

(2) It seems more appropriate to study systems of second degree sentences in two variables when a knowledge of all of the conic sections, and some knowledge of transformations in the plane are available to the student.

(3) Study of systems of equations in the 10 - 12 sequence can arise naturally in the spiral of the "stream" of modeling and linear programming.

Grade 9 - Chapter 4

Measure Functions and Their Properties

1966 Outline, pp. 346-352.

In the outline for Grade 8, Chapter 10, Properties and Mensuration of Geometric Figures, there may be some material that would fit nicely here, particularly the sections using the function notation.

Grade 9 - Chapter 5

Statistics

The 1966 Outline, page 321, only listed some topics that might be included. In the 1966 Outline also, on pp. 417 and 418, there is a statement concerning probability and statistics for Grades 7-9.

The following document was produced to guide the Grades 8 and 9 writing teams. Only the last part is specifically for Grade 9, but the complete document is included here. Hopefully this will assist in getting a consistent sequential treatment in Grades 8 and 9.

See under Grade 8 the document Probability and Statistics, Grades 7, 8, 9, 10 or 11, pp. 40 . This document was, in a sense, superceded by the document Probability and Statistics for Grades 8 and 9, but it still has good ideas in it that should not get lost.

Probability and Statistics for Grades 8 and 9

We question the feasibility of the approach toward probability taken in the Outline and already implemented in Grade 7. Our greatest fear is that the student will not see that a probability model is constructed to reflect a physical situation. More important, the student should understand that the assignment of probabilities, while arbitrary, reflects the physical situation as determined by experiment. Why, in a coin tossing game, is the à priori probability for heads chosen to be $\frac{1}{2}$? What arguments could a 7th grader advance to refute the following argument: If two coins are tossed there are 3 outcomes possible (2 heads, 2 tails, or 1 of each) and that hence each should be assigned the probability $\frac{1}{3}$?

We believe it is more natural to approach Probability and Statistics from the standpoint of statistics and an elementary analysis of a collection of data. The following outline suggests a possible treatment for Grades 8 and 9.

(We do not feel sufficiently competent to provide a detailed outline for the high school course in Probability and Statistics.) Our grand goal for Grade 9 is not so much a command for the calculus of probabilities as it is a feeling for the strengths and weaknesses of assertions like:

1. Toothpaste A is significantly more effective in preventing tooth decay than toothpaste B .
2. Lung cancer can be statistically linked with cigarette smoking.
3. The likelihood of rain today is 60 percent.
4. The average 15 year old boy weighs 105 pounds and is 5'10" tall.
5. Should I purchase a car battery for $20 which may last 18 months or one for $30 which may last 30 months?
6. Two radio signals are heard on the same frequency. One is code from a Russian satellite, the other is noise from outer space. How can we identify which is which?
7. A plastic toy manufacturer uses a machine which unfortunately produces defective toys 10 percent of the time. He is considering buying a new machine at the cost of $10,000 which will produce his toy and which is claimed to have a defective rate of only 5 percent. How should he decide if (a) the new machine has a defective rate less than 10 percent and (b) is it an economical replacement?

GRADE 8.

1. Frequency Distributions.

    1-1. Data from observations where the entire population is known. Select examples which can be developed by the students.

        (1) Heights of class members.

        (2) Distance class members can throw a ball.

        (3) Standing broad jump.

        (4) Test scores.

(5) Birthdays -- by the month.

(6) Number of children in families of class members.

(7) Age of students in months.

(8) Number of letters in last name. (Also first name. Compare.)

(9) Vowel frequencies in newspapers. Compare English and foreign language papers.

(10) Measurement. With a ruler marked in millimeters, let each student measure a line segment of about a yard.

(11) Estimate midpoint of a line segment of about a foot by eye. Then measure the estimates.

(12) Weight of apples (or oranges) in a box.

1-2. Graphs of data. Grouping of data (give rules of thumb). Continuous model?

1-3. Relative frequency. Cumulative frequency.
Raise lots of questions about the properties of the distributions discussed above.

1-4. Mean, Mode, and Median.

Develop as numbers which describe the total distribution; that is, as examples of number valued functions of the set of distributions.

Give different distributions with the same mean.

Percentiles.

Rescaling: If, for example, in the ball throwing experiment, the distances range for 60' to 150' , we could shift the origin so that the range is -45 to 45 or we could rescale so that the range of values is -1 to 1 . Compare a shift of the origin with change of scale of the axis. Try out both on examples in Section 1-1. Contrast scaled and unscaled distributions of Examples 2 and 3 .

1-5. Variance and Standard Deviation.

Treat as further examples of numbers which describe the whole distribution. Compute for the various distributions in Section 1-1. Ask for comments!

Change of scale effect on variance and standard deviation.
Look to Chebychev's inequality but don't emphasize.

1-6. Subpopulations (Samples).

For example, select out distributions for both boys and girls in the examples in Section 1-1. Plot both distributions on the same graph. Compare. Compare with whole. Compute means and standard deviation. Repeat when, for example, the subpopulation consists of those with first initial A - L and with first initial M - Z.

1-7. Samples.

Treat as similar "hunks". Show how some of the examples in Section 1-6 seem to reflect total distribution while others do not. Compare means and standard deviation. Can these sample statistics be used in prediction?

1-8. Measurement -- Distribution of errors.

Approach from the point of view of comparing different groups of measurements of the same object. Example: Have the class measure with a ruler marked in millimeters a line segment about a yard long. Now consider different subpopulations of different size as though they had determined the length of the segment. Compare. Do not try to suggest that an underlying distribution for the errors in measurement might exist. Just treat it as, "This is what we got".

GRADE 9.

1. Examples from Bernoulli trials.

Coin tossing, spinners, thumbtacks. (Pick up link with Grade 7.) Perform say 100 trials of coin tossing 20 times. Compute means and standard deviation of these 20 experiments. Repeat the experiment with different sample sizes than 100 . Compute the means and standard deviations -- relate to size of sample.

2. Probability Models.

Take another look at Grade 7. Treat as modeling problem.

Construct Probability space - Event space.
Assignment of probabilities.

Elementary calculation of probabilities.

Give tables for the Binomial distribution to avoid complications of sophisticated counting.

3. Elementary testing of hypothesis.

The teacher presents statistics from the spins of an unknown spinner. How is the area of the spinner divided?

An ESP experiment with coin tossing: Is the subject doing "significantly" better than he could by guessing? Develop notion of maximum likelihood from the point of view of rejection-acceptance tests. For example, with the spinner problem with a 90 percent confidence we might reject the hypothesis that the distribution was $1/4 - 3/4$ and accept that it was $1/2 - 1/2$. We might also accept a great many others and reject a great many. Which seems most favorable?

Go back and pick up examples from Section 1-6. If a student's performance is given can we decide whether the student is male or female? Suppose, for example, we know the distance of the students standing broad jump. One way of reducing the complicated distribution of distances to a Bernoulli trials situation is to make a pairing of the boys and girls. For each pair, record a 1 if the boy's score exceeds that of the girl and -1 otherwise. Refinements in the method of pairing can bring out other interesting phenomena. For example, is height a key factor in the standing broad jump? Arrange the matched pairs so that they have the same height.

4. Problems requiring more complex computations of probabilities.

Problems where there are more than two outcomes.

ESP experiments where the subject calls the cards from a deck of say 12 cards. When is a long run of successful guesses significant?

Dice (or a suitable euphemism).

Independence of trials.

Selection without replacement.

(In this section we would continue the emphasis to a test of hypotheses. These experiments will lead to the next section where the counting tools are developed.)

5. Combinatorics -- Systematic counting.

   Inclusion - exclusion principle.

   Multinomial coefficients.

   Tree diagrams.

6. Conditional Probabilities.

   Independence of tests.

7. Random Variables.

   Return to Grade 8 type of examples and read off various functions of the distributions. Use Bernoulli trials. Discuss the R. V. which is the number of tosses before the first head.

   Expected value. Expected value of a sum of R. V. is the sum of the expected values. Use this as an aid in determining probabilities and in the combinatorics of Section 2.

8. Correlations between different random variables.

   Curve fitting -- Distinction between best fit and goodness of fit.

Grade 9 - Chapter 6

Displacements - Vectors

It has been suggested that Grade 8, Chapter 3, Displacements, in the 1966 outline be moved here. In the notes for the Grade 8 writing team and the next two pages there is a note of caution about making this move. What goes in this chapter depends upon the final decision on what stays in Grade 8 and also the treatment of vectors in the Grades 10-12 block.

The following documents from the 1966 outline are pertinent to this chapter.

(1) Outline for Grade 8, Chapter 3, Displacements, pp. 207-219.

(2) Outline - Vectors, pp. 434-467.

(3) Vectors on a Line, pp. 424-433.

Grade 9 - Chapter 7

Transformations

1966 Outline, pp. 331-345.

Grade 9 - Chapter 8

Circular Functions

1966 Outline, pp. 357-366.

Grade 9 - Chapter 9

Tangency

1966 Outline, pp. 367-376.

Grade 9 - Chapter 10

Measure

1966 Outline, pp. 377-380.

In the outline for Grade 8, Chapter 10, pp. 20-33, there may be some material that would fit nicely here. The material is a treatment of area, volume, work, and falling body problems without limit processes. The treatment uses double inequalities and has a version without function notation and one using function notation.

Grade 9 - Chapter 11

Complex Numbers

1. Background assumed from prior Grades 7-9 experience.

   (1) Definition of and experience in computing with square roots and the distance formula: Such material is included in Grade 7, Chapter 14, Grade 8, Chapters 2 and 5. We assume that in the definition of square root it has been observed that it is necessary to specify that the square root operation is applied only to nonnegative numbers. Let us assume that this has been sufficiently emphasized and that a question has been raised about possible square roots of

a negative number n and a passing reference made to the existence of complex numbers.

(2) Absolute values: Let us assume here that the geometric role of absolute value in connection with distance on the number line has been discussed.

(3) Commutative, associative, distributive laws; identities and inverses for addition and multiplication; discussion about "structure" of various numbers systems: Such material is now included in Chapter 6, Grade 7. However, it is probably too fancy there for that level and it has been suggested that it be dropped at that level. If it is dropped in its present form it probably should be replaced by a treatment which covers some of the same ground, but with a lighter touch somewhere in the seventh or eighth grade, and reinforced at several points in the seventh and eighth grades. In the process there will surely have been discussion of various extensions of number systems in response to either mathematical needs (e.g., a solution for $x + 7 = 4$) or for more adequate models for "real world" phenomena. After the most recent such extension (to the real numbers) a question about whether this is the last extension possible or necessary will feed into complex numbers in an obvious way.

(4) Solution of quadratic equations; completing the square; possibly the quadratic formula: Here one can avoid quadratic equations without solutions in the real numbers only by being careful in the choice of coefficients. We assume that this fact has been noted -- even emphasized -- and that the question has been raised whether a further extension of the number system would give solutions to such equations. An affirmative answer should be given along with an indication of what such numbers would be, for at least such an equation as $x^2 + 1 = 0$. Probably this should be carried even further and that complex numbers of the form $a + bi$ as results in completing the square or quadratic formula should be explicitly dealt with; though one should probably not develop complex numbers as a system with the field properties at this point. Since quadratic equations are still in the set of materials intended for everybody, as the present chapter probably is not, it would be in order to go on with complex numbers at this point, to point out that they can be

represented as ordered pairs (a,b) and hence can be given tangible existence by representation in Argand diagrams. A passing reference to the usefulness of complex numbers in electrical circuit and other applications should be made, though not in great detail.

(5) Vectors in the plane; the set of ordered pairs as one interpretation of vectors: In addition to setting up the Argand diagram representation of complex numbers we should also pave the way for complex numbers by observing that there is no "closed" multiplication of vectors defined -- that is, there is no multiplication where a vector times a vector gives a vector. The observation that invention of such an operation for two dimensional vectors has something to do with complex numbers then becomes part of the spiral leading to a more grand treatment of complex numbers.

If the things listed above do in fact appear in a reasonable way in the material prior to this final chapter of the ninth grade book then we have probably fulfilled our responsibility to make "everyman" aware of the need for, existence of, and (lightly) usefulness of complex numbers. (For our "everyman" the opportunity should also be taken, in these references to the existence of complex numbers, to include some historical material on how they were invented, their initial reception, and the later discovery that complex numbers are useful in building mathematical models in a number of applications.)

If this is so, then there is a genuine question as to whether there is a need for a chapter at this point before the student has the technical equipment and maturity to deal with the polar form of complex numbers, de Moivre's theorem, and the various lovely uses to which complex numbers in polar form can be put. It is the consensus of the summer 1967 outlining group that such a chapter would nevertheless be useful and that it should take the form of exploiting complex numbers as a way of reviewing and pulling together a lot of previous material about numbers and number systems. With this specification in mind we are suggesting outlines for two versions which cover essentially the same ground from slightly different viewpoints. A brief description of these two versions is included here.

2. Brief Description of the Complex Number Chapter.

Both versions would begin with some discussion of the ways in which and the reasons for which mathematical systems are created and extended. It would be observed that the motivation sometimes comes from the uses of mathematics and sometimes from the mathematics itself -- the latter in response to efforts to resolve paradoxes; fill in apparent gaps; build neat logical systems; and so on. As an example of the former we would have negative numbers as descriptions of, for example, temperatures; as examples of the latter, negative numbers to provide solutions to such equations as $x + 5 = 3$ or names for points to the left of zero on the number line. Both treatments would then have a few paragraphs reviewing the various extensions of the whole numbers carrying along the possible motivation for such extensions from the point of view of uses made of numbers (the things for which the numbers provide models); the need to have solutions for progressively more complicated equations; and the partly geometric problem of naming points on the line. Both would probably gloss over the problem of extending the rationals to the reals (do we intend an honest treatment of this anywhere?) relying at this point mostly on the geometric motivation. Both would probably rely on the algebraic motivation for raising a question about a further extension beyond the reals, namely, are there solutions for all quadratic equations and in particular for the equation $x^2 + 1 = 0$? All this might be done by questioning rather than telling, and would occupy only a few pages at most. The two versions would diverge at this point.

(1) Version 1 of a chapter on complex numbers for the end of the ninth grade.

This would follow the line set out in SMSG Intermediate Mathematics, Chapter 5. Here the observation is made that in each extension of the number system the new system contains the old system as a subset (or is isomorphic to a subset) and this subset must behave in the expected way. One usually demands also that the new system have at least the properties that the old system had. In the extension one must specify what the elements are; one must define equality; one must define addition and multiplication; and one must verify that the newly defined addition and multiplication have the expected properties -- that is the properties of the previous system -- and new properties must be stated explicitly. The new set of elements would then be defined as elements of the form a + bi where a and b are real, with the i having the property that it is a solution for the equation $x^2 + 1 = 0$. One then demands that this set of elements have a multiplication and addition which satisfies certain properties -- the usual properties of the field -- plus the property that $i^2 = -1$. One then rigs the definitions of multiplication and division in such a way that this will indeed be the case and verifies this. The verifications involve one in a number of finger exercises. All the algebra is done with elements of the form a + bi. The details can be checked in Chapter 5 of Intermediate Mathematics. Following this it is observed that a + bi involves the two real numbers a and b and that this suggests the possibility of representing complex numbers as points in the plane. From this the geometry of complex numbers is developed including the absolute value (or modulus); geometric interpretation of the operations on complex numbers; the line and conic sections represented as absolute value equations; and so on. Of course, there is a great deal of manipulation and finger exercises here. It would be nice to have a good simple-minded treatment of how complex number diagrams are useful in electricity for representing phase relationships and how the operations on complex numbers do nice things in these diagrams that serve very well as models for electrical circuit situations.

Brief Outline of Version 1 of Grade 9 - Chapter 11

Complex Numbers

Motivate by solution of $x^2 + 1 = 0$. Trace the development of extensions of the number system from the counting numbers to the reals:

Is there a solution in the set of

counting numbers to $x + 5 = 5$?

whole numbers to $x + 5 = 3$?

integers, etc.

This was probably done before, so now more understanding and appreciation can be expected and the discussion can be on a higher level. The overview is important.

The student must be reminded that each extension was such that the new number system contained the previous one as a subset, with all properties preserved and new ones carefully examined.

At this point summarize the properties of the real number system so that this next extension becomes clear and meaningful:

> If there exists a solution to $x^2 + 1 = 0$, then x^2 must be -1, since the additive inverse is unique.

No real number will meet this condition. So the extension is necessary; we need a number whose square is -1.

At this point then, the new number system will contain all of the reals and at least one more element: it is called i and defined such that $i^2 = -1$. Any other elements needed? Etc.

Since this is for 9th graders, the treatment in Intermediate Mathematics, Chapter 5, SMSG, is a bit hard in spots, but can be adjusted.

Objection was raised to calling i a new symbol, since i^2 is a real number -- I tried to meet this objection above.

Also, if Chapter 5, SMSG Intermediate Mathematics is used, change the sequence of the sections: 5-1, 5-2, 5-3, 5-4, or parts thereof, followed by 5-7, 5-5, 5-6, etc. This will introduce the graphic representation earlier.

(2) Version 2 of a chapter on complex numbers for the end of the ninth grade.

This version differs from the first version principally in the way it handles the various extensions of the number systems. It departs from the student's previous experience by doing each extension (except from the rationals to the reals) by means of an algebra on ordered pairs. This way of doing things is familiar to us and we see it as rather neat; there is some question as to whether the ninth grader would also see it as neat and worth the trouble. One would begin not with Peano postulates and development of the properties of natural numbers by induction but rather take as already known the natural numbers, multiplication and addition of them, and the CAD and identity properties. One then develops an algebra for integers with pairs (a,b) -- where (a,b) is to be interpreted as a - b -- invents the kinds of operations that would work for the a - b interpretation, defines equality and verifies that the usual properties hold and that in addition there is now closure for subtraction. In the process one must deal already with equivalence classes whereas in the conventional treatment this does not become necessary until you have rational numbers. One then takes the set of integers as known and develops the set of rationals as ordered pairs (a,b) -- where (a,b) is now interpreted as a/b -- and again defines equality, invents the operations, and verifies that the usual properties hold and that now we have closure for division. The interesting thing here is that whereas for the integers multiplication was the complicated operation to define, here addition is the more complicated. The extension of the rationals to the reals cannot, of course, be done with ordered pairs; nor can we define an operation that is implicitly an algorithm in the same way that we can for the other sets of numbers. We do this presumably as honestly here as we do anywhere -- the details are not quite clear. With reals in hand the complex numbers are developed by ordered pairs of real numbers (a,b) -- where (a,b) is now interpreted as a + bi , with i as the number with the property $i^2 = -1$ familiar from previous work -- and the operations invented and properties verified in a way entirely parallel to the previous extensions. Since complex numbers correspond to ordered pairs the geometric representation is immediate and much the same work with this geometry is carried out as is the case in the first version.

This may be far too fancy for the end of the ninth grade, but some of us feel that this novel way of getting at the various extensions would be fairly exciting to at least a certain number of high school youngsters. It has the further advantage of making the complex numbers another extension of the number system much like a couple of previous ones, rather than making them the unnatural things suggested by the words "imaginary" or "complex".

3. Suggestions for Developing Geometric Interpretations of Complex Numbers.

These are applicable to either Version 1 or Version 2.

(1) Graphic Representation of Complex Numbers.

Associate with $a + bi$ the point (a,b) in the plane.

Example: Give the coordinates of the points association with

(a) $Z_1 = 2 + 3i$ graph and observe

$Z_2 = 4 + 6i$

$Z_3 = -2 - 3i$

(b) $Z_1 = 1 + 3i$ graph and observe

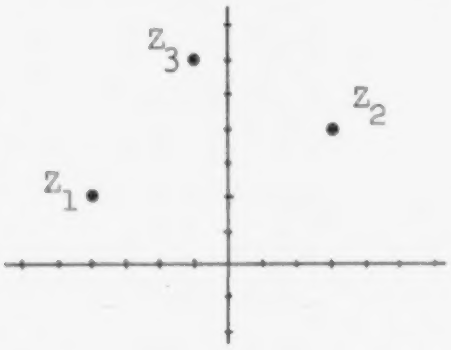
$Z_2 = 2 + i$

$Z_3 = 3 + 4i$

(c) let $Z_1 = -4 + 2i$

$Z_2 = 3 + 4i$

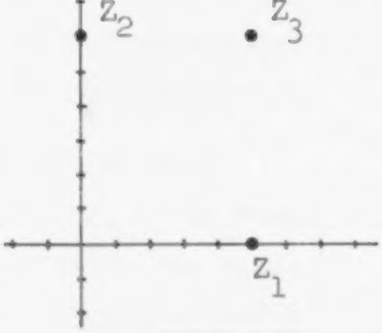
$Z_3 = -1 + 6i$



(d) let $Z_1 = 6 + 0i$

$Z_2 = 0 + 5i$

$Z_3 = 6 + 5i$



(e) Draw quadrilateral ABCD with A(-2,5) , B(3,2) , C(0,0) , D(-5,3) ; Z_1 is associated with D , Z_2 with B ; find $Z_1 + Z_2$.

Continue with similar examples and develop rule for addition.*

(2) Absolute Value of Complex Numbers.

How far are 3 , -3 , 5 , -5 , $\sqrt{7}$, and $-\sqrt{7}$ from the origin?

Distance is measured by nonnegative numbers. The distance between a real number n and the origin was defined as $|n|$.

What is the distance between $Z_1 = 2 + 3i$, and the origin?

Draw it. Develop result:

$$|Z_1| = \sqrt{a^2 + b^2}$$

Example: Give the absolute value of the complex numbers of Example (a) - (e).

Given $|Z| = 5$; can you find Z ? Is there only one complex number whose absolute value is 5 ?

If not, give others.

If there are several, can you describe their position?

More examples.

(3) Multiplication as Effecting a Rotation.

Draw, in the complex plane (1,0) , (0,1) , (-1,0) , and (0,-1) .

$$(a)\begin{cases} 1 + 0i = 1 \\ 0 + 1i = i \\ -1 + 0i = -1 \\ 0 - 1i = -i \\ 1 + 0i = 1 \end{cases}$$

---

*Verbalize -- compare with vector addition.
If vectors were studied before, of course, addition should or can be done "vectorially".

(b) Multiply

$(1 + 0i)i = ?$

$(0 + 1i)i = ?$

$(-1 + 0i)i = ?$

$(0 - 1i)i = ?$

graphically, what does multiplying by i mean?

(c) $Z_1 = 2 + 3i \rightarrow (2,3)$

$$Z_2 = Z_1 \cdot i = (2 + 3i)i$$
$$= -3 + 2i \rightarrow (-3,2)$$

Draw the graph

Any relation to (b) above?

$$Z_3 = (-3 + 2i)i$$
$$= -2 - 3i$$

does the same observation hold?

If $A(a,b)$ and $B(-b,a)$, what is the relation between $\overline{OA}$ and $\overline{OB}$? Between $a + bi$ and $-b + ai$?

Multiply $a + bi$ by any real number, c.

$(a + bi)c = ac + bci$; what does this mean graphically?

Example: Choose 3 complex numbers, multiply each by 2, -1, 3, in turn. What do you observe?

Can we now multiply, graphically, two complex numbers?

(a) Let $Z_1 = 2 + 3i$ and $Z_2 = -1 + 2i$, using the distributive property,

$$\begin{aligned} Z_1Z_2 &= (2 + 3i)(-1 + 2i) \\ &= (2 + 3i)(-1) + (2 + 3i)(2i) \\ &= -(2 + 3i) + 2(2 + 3i)i \\ &= -Z_1 + 2i \cdot Z_1 \end{aligned}$$

From the preceding exercises we know how we can represent graphically

(a) $-Z_1$,

(b) $2iZ_1$,

(c) $-Z_1 + 2iZ_1$ hence Z_1Z_2.

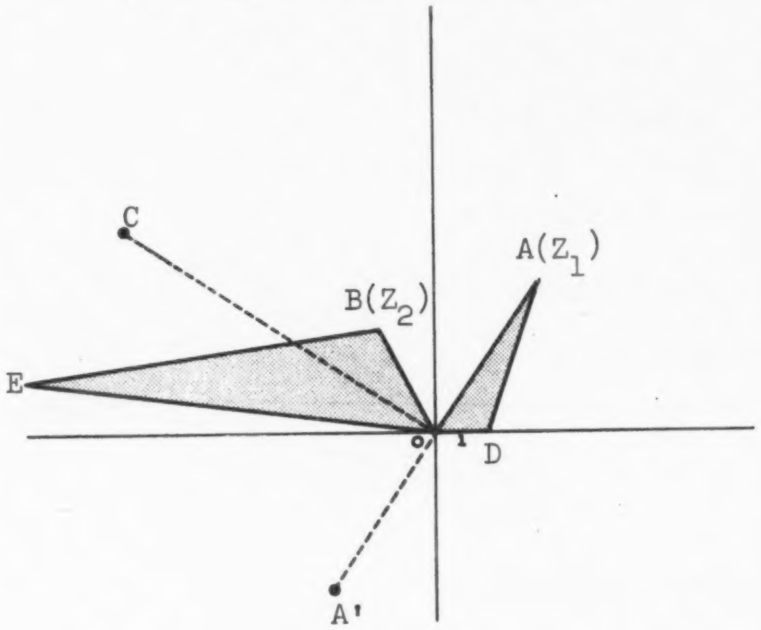
Associate Z_1 with point A , Z_2 with point B , $-Z_1$ with point A' , $2iZ_1$ with point C . Then the coordinates of the points A , B , A' , and C are respectively (2,3) , (-1,2) , (-2,-3) , and (-6,4) . Now let Z_1Z_2 be associated with E ,

$$Z_1Z_2 = -Z_1 + 2iZ_1$$

E has coordinates (-8,1) , therefore

$$Z_1Z_2 = -8 + i .$$

Now compare $\triangle OAD$ and $\triangle OEB$.



$OD = 1$

$AD = \sqrt{10}$

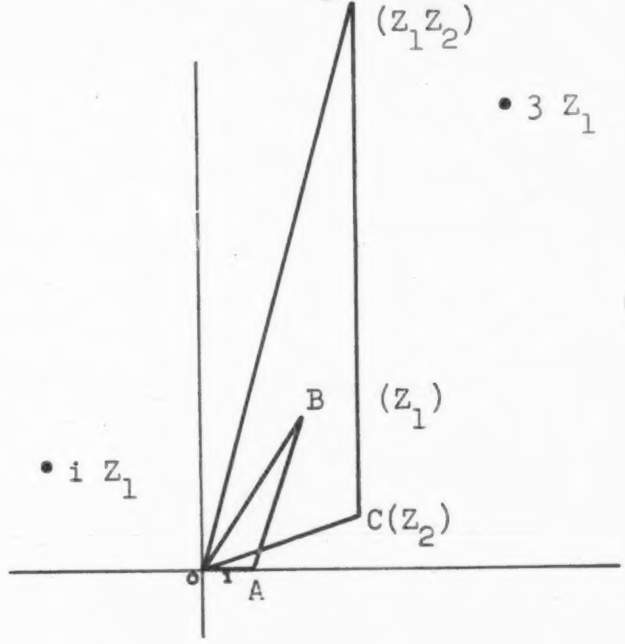
$OA = \sqrt{13}$

$OB = \sqrt{5} = 1 \cdot \sqrt{5}$

$EB = \sqrt{50} = \sqrt{10} \cdot \sqrt{5}$

$OE = \sqrt{65} = \sqrt{13} \cdot \sqrt{5}$

(4) Construction of Z_1Z_2 Using Similar Triangles.



$Z_1 = 2 + 3i$

$Z_2 = 3 + i$

Multiply graphically, using first the distributive property:

$$(2 + 3i)(3 + i) = (2 + 3i)3 + (2 + 3i)i$$

$$= (2 + 3i)3 = 3Z_1$$

$$(2 + 3i)i = Z_1 \cdot i$$

$$Z_1Z_2 = 3Z_1 + iZ_1$$

$Z_1 \rightarrow (2,3)$
$3Z_1 \rightarrow (6,9)$
$iZ_1 \rightarrow (-3,2)$
$3Z_1 + iZ_1 \rightarrow (3,11)$

$\triangle OAB \sim \triangle OCD$

$OA = 1 \qquad OC = \sqrt{10} = 1 \cdot \sqrt{10}$
$AB = \sqrt{10} \qquad CD = 10 = \sqrt{10} \cdot \sqrt{10}$
$OB = \sqrt{13} \qquad OD = \sqrt{130} = \sqrt{13} \cdot \sqrt{10}$

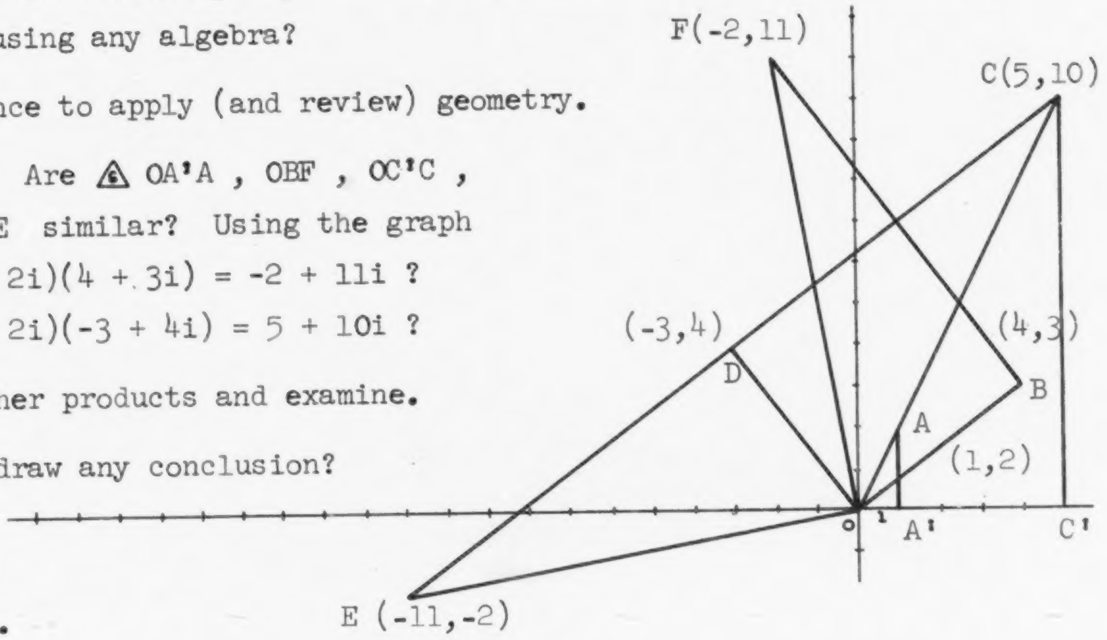
After one or two examples, can the similar triangles be constructed without using any algebra?

Good chance to apply (and review) geometry.

Example: Are $\triangle$ $OA'A$, OBF , $OC'C$, ODC , ODE similar? Using the graph is $(1 + 2i)(4 + 3i) = -2 + 11i$? is $(1 + 2i)(-3 + 4i) = 5 + 10i$?

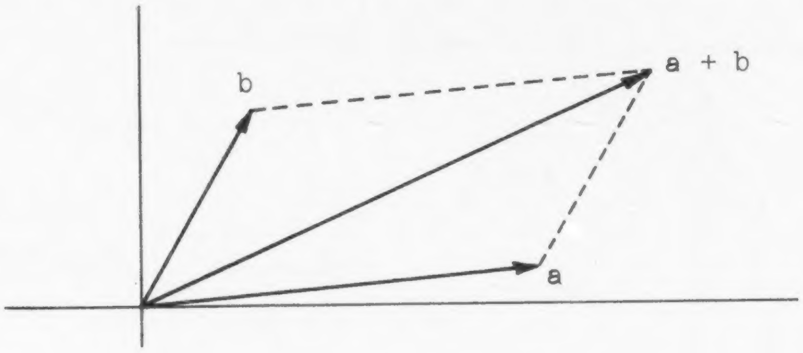
Write other products and examine.

Can you draw any conclusion?



(5) Addition.

(a) Add complex numbers "vectorially".



(b) Triangle inequality.

$|a + b| \leq |a| + |b|$ (Triangle a , b , (a + b))

$|a - b| \geq \big||a| - |b|\big|$

(6) Multiplication.

(a) $|ab| = |a||b|$

(b) DeMoivre's Theorem

(7) Exercises.

Find all points z such that

(a) $|z - a| \leq r$ (Disk)

(b) $|z - a| = |z - b|$ (Line)

(c) $|z - a| \leq |z - b|$ (Half-plane)

(d) $|z - a| + |z - b| \leq 2r$ (Elliptical region)

(e) $|z - a| + |z - b| \geq 2r$ (Exterior of ellipse)

(f) Gaussian Integers. $G = \{a + bi : a , b \text{ integers}\}$
Find the units: $\pm 1 , \pm i$
Show that there is a euclidean algorithm:
For every pair of nonzero Gaussian integers w, z there exist Gaussian integers x , y such that $z = wx + y$ with $|y| < |w|$.

Interpret geometrically:

Since $|z/w - x| = |y/w| < 1$ the assertion is equivalent to saying that within a circle of radius 1 from any point (z/w) there is a complex number $x = a + bi$ with integral a and b .

Show that any two Gaussian integers have a greatest common divisor.

(8) Transformations of the Complex Plane.

Consider functions of complex numbers into the complex numbers of the form

(a) $f(z) = z + a$ (Front cases with a real, and complex)

(b) $f(z) = az$

(c) $f(z) = z^2$

(d) $f(z) = \frac{1}{z}$

(e) $f(z) = \frac{1}{z - i}$

(f) $f(z) = \frac{1}{z^2 + 1} = \frac{1}{2i} \left[\frac{1}{z - i} - \frac{1}{z + i} \right]$

In each example explore geometrical questions.

How does each function transform a line, a circle, a square, an angle?

Be sure to treat the cases where the line "goes through" the constants in the definition of the function.

Perhaps the most interesting case to study is $f(z) = \frac{1}{z}$. Here points inside the unit circle go outside and conversely. Lines go into circles, and conversely.

The Flexibility of the Grades 10-12 Outlining

The 7-9 sequence was designed to develop wide-range mathematical content for the generally educated population. Within the 10-12 sequence, there is less certainty about what content will be of specific value or is even essential in developing mathematical maturity. In fact, it is recognized that several quite different sequences might easily be of equal value in maintaining interest and in developing mathematical maturity. There may well be several "royal roads" depending upon the particular group of students. It is the thought of the outlining group that flexibility of content after the 7-9 sequence is essential for keeping the largest possible number of students enrolled in mathematics and doing mathematics successfully.

Any 10-12 curriculum should allow for many diverse student groups: (1) the mathematically oriented, college-bound students who need the potential of completing a substantial amount of calculus while still in high school; (2) the mathematically capable and interested students who may progress somewhat more slowly but can develop a significant mathematical background if they are not forced along too quickly; (3) the mathematically capable students who are not mathematically oriented but will probably need a breadth of mathematical ideas to help within their areas of interest; (4) the non-directed, semi-successful students who need further time, involvement and exposure until they find an area of interest and/or ability. Each of these groups can probably benefit from further work in mathematics if it is of mathematical worth, if it interests them, if it is developed at a pace they can handle, and if it is within their intellectual grasp.

A lot of talk and thought has been given by the outlining group to the possibilities of several quite different sequences for 10-12, depending upon the students' (and teachers') background, ability, interests, and future plans. It is still the thought of most of the group members that the outlining suggestions contain this potential. Specifically, the attached possible sequences are envisioned.

The courses used within the sequences are described in other documents. Briefly, they are:

1. *Axiomatic/Deductive Block (1 semester)
2. Vector-Analytic Geometry and Functions (1 semester)
3. Vectors and Linear Algebra (1 semester)
4. Elementary Functions and Calculus (2 semesters)
   (minimal for AP level 1)
5. Advanced Placement Calculus (3 semesters)
   (AP level 2, including one semester of Advanced Elementary Functions)
6. Probability and Statistics (1 semester)
   (1) With calculus
   (2) Without calculus
7. Computational Mathematics (1 semester)
   (Elementary, non-calculus, computer-oriented)

Note: The course marked with an asterisk is open to several selections of particular content; see Grades 10-12, Deductive Block, pp. 121.

The boundary conditions used in making these sequences are:

(1) The background for these sequences is completion of the 7-9 curriculum.

(2) The maximum time available is 6 semesters. There may be fewer semesters available (for those who take a longer time to complete the 7-9 sequence) but probably not more.

(3) The suggested time allotments are minimal. It may be necessary to stretch any given course or courses over a longer time span; e.g., Vector-Analytic Geometry and Functions, and Vectors and Linear Algebra may be done in 2 or 3 semesters. The emphasis should be completion of a content block rather than completion of a standard block of time.

The following pages contain two versions of the various possible sequences: one is a two-page diagram that gives a concise overview and the other is a five-page description with titles and possible time allotments. The suggested sequences are the same. The "flexibility factor" should be considered from two different aspects.

First, there is a flexibility of scheduling. It is not contemplated that any one school will offer all of these sequences (or even half of them). Upon selection of one or two of these sequences, it will be noted that the basic courses may be scheduled in an order and with sufficient frequency to allow for student cross-over and section cross-over. That is, the basic modules may often be fitted within several somewhat different sequences.

Actually there are essentially only two fundamental three-year sequences -- one leads up to a full year of advanced placement calculus; the other to a year of elementary function and calculus (1/2 year advanced placement calculus). A given school system may decide that it can offer both of the fundamental sequences or only one. The other sequences listed in the blocks in the first row are obtained by changing the time at which the Deductive Block is given and the time at which Probability and Statistics is given. Once these two decisions are made, the shorter sequences are generally obtained by omitting some of the courses in the fundamental sequence -- usually ones at the end of the sequence. In addition, some of the shortened sequences are completed with a course in Computational Mathematics.

Second, there is a flexibility for the individual student's programs. Through the use of different blocks, it is quite possible to establish final goals different from the calculus. On these sequences, there is an additional three-way time factor flexibility: (1) the student may be beginning the 10-12 sequence later than the 10th grade to allow for completion of the 7-9 material; (2) the fewer number of content blocks may be extended in time to allow for slower moving students; (3) the student may not elect to continue mathematics for a full three more years.

It is hoped that several different orderings of courses and student programs will be tried in different schools and some evaluation made to support or discount the envisioned flexibility and mathematical worth.

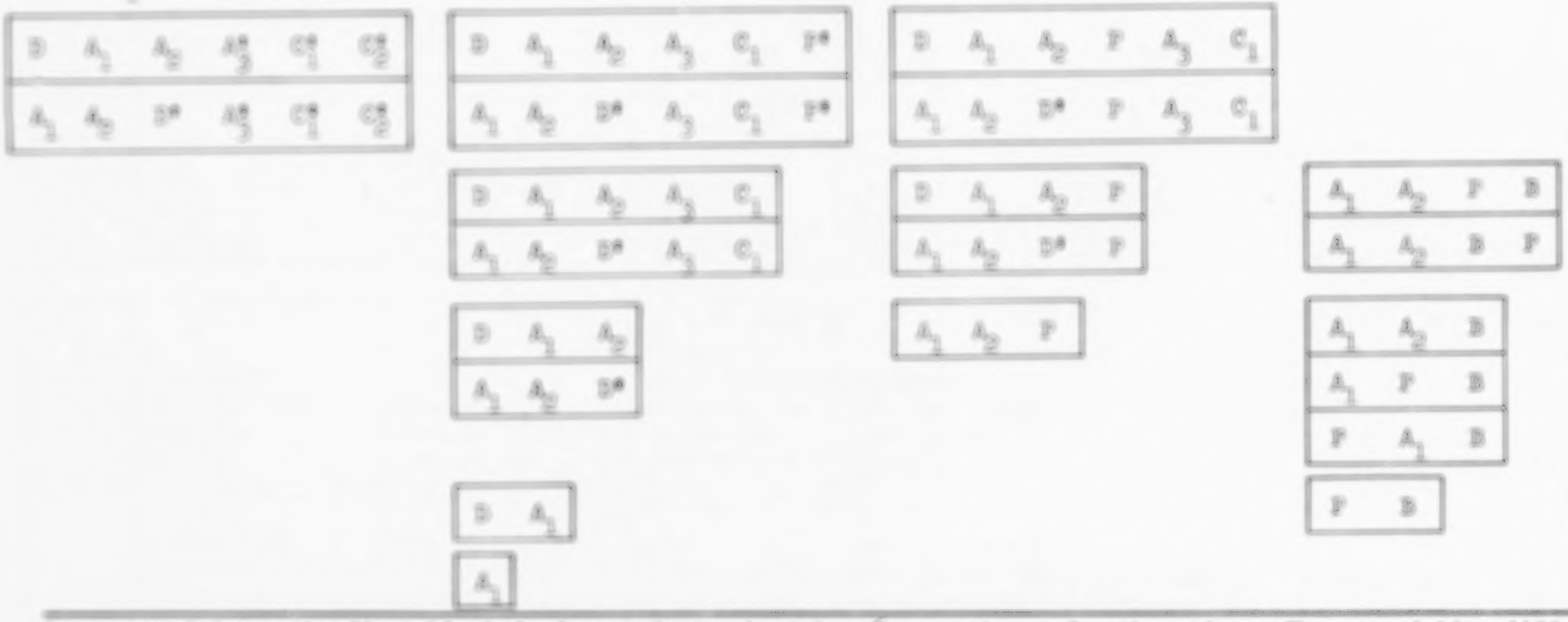
Two courses, Probability and Statistics (one semester) and Computational Mathematics (one semester) have both been included in this document. The group has not outlined the content of either of these two courses. There are a number of schools that offer such one semester courses right now, using presently available texts. It is felt that the content of these two courses should be somewhat different when they are fitted into the new sequence of courses presently being outlined and written.

Flexible Sequences - Version 1

ALL COURSES ARE SEMESTER COURSES

KEY:

D	Deductive Block - Synthetic Geometry
D^*	Deductive Block - Synthetic Geometry Point Set Topology Math. Systems Number Theory, etc.
A_1	Vector-Analytic Geometry and Functions
A_2	Vectors and Linear Algebra
A_3	First Semester, Elem. Fns. and Calculus
C_1	Second Semester, Elem. Fns. and Calculus
A_3^*	Preparation for Adv. Pl. Calculus (Calculus BC)
C_1^*	First Semester, Yr. of Adv. Pl. Calculus (Calculus BC)
C_2^*	Second Semester, Yr. of Adv. Pl. Calculus (Calculus BC)
P	Probability and Statistics
P^*	Probability and Statistics with Calculus
B	Computational Mathematics (Elementary, non-calculus, computer-oriented)



D A_1 A_2 A_3^* C_1^* C_2^*
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D A_1 A_2 A_3 C_1
A_1 A_2 D^* A_3 C_1

D A_1 A_2
A_1 A_2 D^*

D A_1

A_1

D A_1 A_2 P A_3 C_1
A_1 A_2 D^* P A_3 C_1

D A_1 A_2 P
A_1 A_2 D^* P

A_1 A_2 P

A_1 A_2 P B
A_1 A_2 B P

A_1 A_2 B
A_1 P B
P A_1 B

P B

As you read down, the first block is for students who take 6 semesters of mathematics. The second line differs from the first only in the order in which courses are taken. The second block is for students who take 4 or 5 courses (they may take more than 4 or 5 semesters to complete the courses). Again the second line differs from the first only in the order in which the courses are taken. The third block is for students taking 3 courses, etc. As you read down each column, some course(s) has(have) been dropped from the preceding block.

Flexible Sequences - Version 2

Track	Semester 1	Semester 2	Semester 3	Semester 4	Semester 5	Semester 6
1	Synthetic Geometry (Deductive)	Vector - Analytic Geometry and Functions	Vectors and Linear Algebra	Advanced Elem. Fns. (Prep. for Adv. Placement Calc.)	Calculus (BC) (1st Sem.)	Calculus (BC) (2nd Sem.)
2	Vector Analytic Geometry and Functions	Vectors and Linear Algebra	Syn. Geom., or Pt. Set Topology, or Math. Systems or Number Theory, etc.	Advanced Elem. Fns. (Prep. for Adv. Place. Calculus)	Calculus (BC (1st Sem.)	Calculus (BC) (2nd Sem.)
3	Synthetic Geometry (Deductive)	Vector Analytic Geometry and Functions	Vectors and Linear Algebra	Elem. Fns. and Calculus (AB) (1st Sem.)	Elem. Fns. and Calculus (AB) (2nd Sem.)	Prob. and Stat. (Using some Calculus)
4	Vector Analytic Geometry and Functions	Vectors and Linear Algebra	Syn. Geom., or Pt. Set Topology, or Math. Systems or Number Theory, etc.	Elem. Fns. and Calculus (AB) (1st Sem.)	Elem. Fns. and Calculus (AB) (2nd Sem.)	Prob. and Stat. (Using some Calculus)
5	Synthetic Geometry (Deductive)	Vector Analytic Geometry and Functions	Vectors and Linear Algebra	Prob. and Stat. (No Calculus)	Elem. Fns. and Calculus (AB) (1st Sem.)	Elem. Fns. and Calculus (AB) (2nd Sem.)

Flexible Sequences - Version 2

Track	Semester 1	Semester 2	Semester 3	Semester 4	Semester 5	Semester 6
6	Vector Analytic Geometry and Functions	Vectors and Linear Algebra	Syn. Geom., or Pt. Set Topology, or Math. Systems, or Number Theory, etc.	Prob. and Stat. (No Calculus)	Elem. Fns. and Calculus (AB) (1st Sem.)	Elem. Fns. and Calculus (AB) (2nd Sem.)
7	Synthetic Geometry (Deductive)	Vector Analytic Geometry and Functions	Vectors and Linear Algebra		Elem. Fns. and Calculus (AB) (1st Sem.)	Elem. Fns. and Calculus (AB) (2nd Sem.)
8	Vector Analytic Geometry and Functions	Vectors and Linear Algebra		Syn. Geom., or Pt. Set. Topology, or Math. Systems, or Number Theory, etc.	Elem. Fns. and Calculus (AB) (1st Sem.)	Elem. Fns. and Calculus (AB) (1st Sem.)
9	(Completion of 7-9 Sequence)	Synthetic Geometry (Deductive)	Vector Analytic Geometry and Functions	Vectors and Linear Algebra		Prob. and Statistics (No Calculus)
10	(Completion of 7-9 Sequence)	Vector Analytic Geometry and Functions	Vectors and Linear Algebra		Syn. Geom. or Pt. Set Topology, or Math. Systems, or Number Theory, etc.	Prob. and Statistics (No Calculus)

Track	Semester 1	Semester 2	Semester 3	Semester 4	Semester 5	Semester 6
11	(Completion of 7-9 Sequence)	Vector Analytic Geometry and Functions	Vectors and Linear Algebra		Prob. and Stat. (No Calculus)	Computational Mathematics (Elem., Non-Calculus, Computer-Oriented)
12	(Completion of 7-9 Sequence)	Vector Analytic Geometry and Functions	Vectors and Linear Algebra		Computational Mathematics (Elem., Non-Calculus, Computer-Oriented)	Prob. and Statistics (No Calculus)
13	Completion of 7-9 Sequence		Synthetic Geometry (Deductive)	Vector Analytic Geometry and Functions	Vectors and Linear Algebra	
14	Completion of 7-9 Sequence		Vector Analytic Geometry and Functions	Vectors and Linear Algebra		Syn. Geom., or Pt. Set Topology, or Math. Systems, or Number Theory, etc.
15	Completion of 7-9 Sequence		Vector Analytic Geometry and Functions	Vectors and Linear Algebra		Prob. and Statistics (No Calculus)
16	Completion of 7-9 Sequence		Vector Analytic Geometry and Functions	Vectors and Linear Algebra		Computational Mathematics (Elem., Non-Calculus, Computer-Oriented)

Track	Semester 1	Semester 2	Semester 3	Semester 4	Semester 5	Semester 6
17	Completion of 7-9 Sequence		Vector Analytic Geometry and Functions		Prob. and Statistics (No Calculus)	Computational Mathematics (Elem., Non-Calculus, Computer-Oriented)
18	Completion of 7-9 Sequence		Prob. and Statistics (No Calculus)	Vector Analytic Geometry and Functions		Computational Mathematics (Elem., Non-Calculus, Computer-Oriented)
19	Completion of 7-9 Sequence			Synthetic Geometry (Deductive)	Vector Analytic Geometry and Functions	
20	Completion of 7-9 Sequence				Vector Analytic Geometry and Functions	
21	Completion of 7-9 Sequence				Prob. and Statistics (No Calculus)	Computational Mathematics (Elem., Non-Calculus, Computer-Oriented)

Polynomial Algebra

Preamble:

Since the first Summer Outlining group "neatly" sidestepped the problem of how to deal with polynomials, we are faced with this decision as to how to proceed this summer. My recommendation is as follows:

1. A "simple-minded" approach to the whole business.
2. Rather than trying to select one approach, (i.e., polynomials considered as functions, or "forms" or as "expressions" as illustrated on pp. 60-67 of the New Orleans Report), I feel that we should use whatever interpretation seems to be appropriate for the task at hand, explain the differences in interpretation, where possible, without becoming "heavy-handed", and let the student develop the same kind of freedom that mathematicians exhibit. (See pp. 65-66 of the New Orleans Report.)

Polynomial Algebra:

A group of "finger exercises, skills, and concepts" to be presented in a block and/or scattered appropriate throughout the tenth grade.

Background:

In Grades 7-9 (or the first three subdivisions of the Outline), the student has been exposed (rather informally in some cases) to the following kinds of polynomials, and operations with these polynomials: ax , $ax + b$, $ax + bx$, $ax^2 + bx + c$. Operations such as simplification, factoring, the solution of equations (completing the square, factoring, and formula) for quadratic polynomials, and graphs of functions related to these polynomials have also been introduced.

Purpose:

The purposes of this section are as follows:

1. To define polynomials and rational expressions.
2. To review, briefly, operations with first and second degree polynomials in one variable.

3. To develop "mechanical" skill in operations with polynomials and rational expressions.
4. To develop graphically the representation of polynomial functions and rational functions.
5. To develop the Remainder, Factor, Location theorems and Descartes's rule of signs for polynomial functions.
6. To develop some skill in working with families of curves and parametric equations.

Rationale:

At this point, the students have been exposed to a considerable amount of what is contained in the present First Course in Algebra. However, the student has only had experience with first degree polynomials in one and two variables, and second degree polynomials of the form $ax^2 + bx + c$, $a^2 - b^2$, $a^2 \pm 2ab + b^2$. A reasonable amount of review and extension of ideas and skill in operations with these and other polynomials seems to be needed in order to facilitate the development of analytic geometry and linear algebra. It is felt at this time that factoring polynomials over different domains, including complex numbers, long division, and the like can be included with the necessary "finger" exercises to provide the skill level needed at this point. In addition it is recommended that the graphical representations of polynomials be extended beyond the previously graphed polynomials $y = ax + b$ and $y = ax^2 + bx + c$. This would include the development of the Remainder, Factor, and Location theorems and a discussion of parametric equations and families of curves. The rationale here being that such an organization would tend to give the treatment of polynomial and rational functions a good reason for existing and, it is hoped, provide a natural bridge to the subsequent spiral treatment of some of these concepts in Analytic Geometry and Linear Algebra sections.

Section 1. Definitions.

1-1. Polynomials are expressions like

$$2x + 3y$$

$$3x^2 + 2y - 5z^9$$

$$\sqrt{5}\, x^3 + \frac{2}{3}$$

$$3x^5 + \frac{1}{2} x^4 - 5x^3 - 7x^2 - 3x + 9$$

$$-5$$

$$\vdots$$

We can form polynomials by combining the elements of two sets using only the following operations of addition, subtraction and multiplication.

The two sets we can use are the set of Real Numbers, or one of its subsets, and a set of variables. We can combine these elements with only a finite number of the operations listed above.

Exercises and Examples: Should involve simple recognition exercises about expressions that are or are not polynomials. Also some exercises where the students are given some elements, and then allowed to construct some polynomials. Develop degree of polynomial.

If the set of numbers used to form the polynomial is the set of integers, then we say that the expression is a polynomial over the integers.

Examples: . . .

Similarly we can have:

(1) polynomials over the rational numbers. (Examples)

(2) polynomials over the real numbers. (Examples)

(3) Exercises.

1-2. Functions like $f: x \rightarrow ax + b$, $g: x \rightarrow ax^2 + bx + c$,

$$h: (x,y) \rightarrow 2x^2 - 3y^2 + 9 \text{ , etc.}$$

which are defined by polynomials are important, and in particular, polynomials in one variable define functions which are models of many situations in engineering, the natural and social sciences, and in mathematics itself.

(Exercises and examples developing recognition of polynomial functions, their degree and evaluation of polynomial functions.)

1-3. Because of the role of polynomials of one variable in mathematics, it becomes necessary to find the sum, product, and analyze the characteristics of such polynomials. (Review briefly what has been done for $f: x \to ax + b$, $g: x \to ax^2 + bx + c$ and then extend examples and eventually define $f: x \to a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$)

Section 2. Operations with Polynomials.

2-1. Review briefly polynomials and factoring over domains -- extend present F.C.A. Chapter 12 to include $x^n \pm y^n$.

2-2. Review briefly addition, subtraction of polynomials.

2-3. Develop division algorithm for polynomials.

Section 3. Algebra of Rational Expressions. (F.C.A. Chapter 12)

3-1. Review and develop some skills with expressions involving exponents, $\left(\frac{xy^{-3}}{5x^{-2}y^{2}}, \dots \right)$

3-2. Multiplication, Addition, Subtraction, Simplification of Rational Expressions.

Section 4. Graphs of Polynomial Functions.

4-1. Review briefly graphs of $f: x \to ax + b$, $f: x \to ax^2 + bx + c$.

4-2. Graph of $f: x \to x^n$, ... , (n even, n odd) .

(Introduction of families of curves, parametric equations.)

Section 5. Remainder and Factor Theorem.

Section 6. Zeros of Polynomial Functions.

Decartes's Rule of signs.

Section 7. Graphs of Rational Functions.

Section 8. I think we also need to be sure that students have some experience in finding solution sets of sentences like:

$$\sqrt{2-x} + 2 = 7 \ ,$$

$$\frac{3x + 5}{x-2} + \frac{5}{x-3} = 10 \ ,$$

etc.,

and perhaps some skills in operations with algebraic expressions like:

$$\frac{\sqrt{x} + 1}{\sqrt{x} - 1} \ , \ \frac{2}{\sqrt{y-7}} + \frac{5}{\sqrt{y-2}} \ , \ \ldots$$

Grades 10-12 Deductive Block

The group felt strongly that such a block belonged in the sequence 10-12 somewhere. It soon became clear that there is some flexibility about where it should appear in the sequence. It was not so clear as to the precise content of this block.

One version went through at least three revisions and that will be given first, followed by a comment supported by some of the members of the group. This version is called "The One Unit Axiomatic/Deductive Block".

The other version is entitled "(Synthetic) Geometry of the Plane", and much of it appears in various forms in other papers. This version is followed by several comments supported by some of the members of the group.

The following is the first version and a comment on it.

The One Unit Axiomatic/Deductive Block

Part 1:

The outlining committee felt that this is an area where some highly varied experimentation is in order. That is, several significantly different approaches should be developed and tested. Even the placement of this block within the curriculum might vary -- either immediately following the 7-9 curriculum or following the Analytic/Vector Geometry, Linear Algebra unit. The intended goal of the experimentation should not be the eventual selection of only one of these possibilities but could lead to the adoption of several (tested and proven) blocks depending upon:

(1) student population -- the different approaches will vary in sophistication, abstraction, and/or maturity requirements;

and

(2) teacher preference -- it is recognized that the teacher will do the best job with an approach in which he feels some security and interest.

Part 2:

The outlining committee was in agreement that each approach should fit within these general guidelines:

(1) There *is* a need for one concentrated unit (covering at most one semester) on material of a primarily deductive nature which will provide the student with the opportunity to develop a knowledge and appreciation of "proof".

(2) The unit should contain at least one explicitly stated axiomatic system as well as a deductive development.

(3) This unit is to be essentially free of coordinates and real number axiomatics.

(4) The approach should develop enough power to allow the student to cope with "originals" at some time in the unit.

Part 3:

The outlining committee makes the following suggestions for quite different approaches to this block but does *not* intend to rule out other possibilities that meet the above stated guidelines. In fact, it is hoped that other suggestions will be made.

Type A - A Geometry Block

Rationale: A number of significant reasons support such a course:

(1) It is a part of the world's cultural and intellectual history. Its words and spirit have permeated our language: diametrically opposed, tangentially related, going off on a tangent, an obtuse mind, an acute observation, parallel development, Q.E.D.

(2) It is an interesting course. Its "originals" have intrigued good students for centuries, and have frequently been a recognizable first point in the development of mathematicians of the past and present.

(3) It is psychologically satisfying. It has a great number of attainable goals. "I solved it." There is a clear sense of "closure" ... you know when you have a solution or you don't.

(4) It is esthetically gratifying. "There is a beautiful proof for that." "Euclid alone has looked on beauty bare."

(5) It is perhaps the best example of an extensive deductive system, with many non-trivial proofs and conclusions. It is a model for such deductive systems. (Spinoza's Ethics)

(6) Its content is more familiar to teachers who would therefore feel more willing and competent to teach the course.

One recommended freedom in constructing a course in the spirit of the above-stated guidelines is in using various sorts of technical machinery:

(1) Some version of the Birkhoff axiom scheme such as is used in the present SMSG Geometry.

(2) A synthetic approach using Euclidean transformations.

(3) An approach through affine geometry and then a specialization to Euclidean. (As per Levi, Elements of Geometry and Trigonometry)

(4) And so on ...

A second recommended freedom within the spirit of the guidelines could be based on various choices of content, both for emphasis and coverage:

(1) Focus on axiom systems and deduction per se, with the geometric content as a vehicle. One would probably not deal with such technical matters as completeness, independence, or categoricity in detail but might deal with the consequences of alternative choices of key axioms; e.g., the various parallel postulates. One would not be concerned with "coverage" of any large amount of content. This approach could lead to some study of various geometries.

(2) Focus on organizing the somewhat random and intuitive experience in geometry contained in the previous three years' work. Here a strong enough set of axioms might be selected and presented so that the work would be largely a matter of fitting quite a lot of content into a sequential structure.

(3) Focus on a rather small portion of Euclidean geometry with some care and in some detail. For example, one might develop the topic of circles after presenting an adequate set of axioms and basic theorems. This particular choice could entail work with congruence, similarity,

parallelism, arc length, angle measure, and notions of tangency. Other such portions of geometry are also rich in possibilities.

(4) And so on ...

Type B - A Non-geometry Block

Rationale: The suggestions under Type B frankly take the point of view that (1) sufficient geometrical content has been included in 7-9 to provide a sound basis for future mathematics, and (2) the course in analytic/vector geometry, linear algebra is a natural successor to the 7-9 sequence. What then remains is to deepen each student's mathematical maturity and further nurture his mathematical creativity, insight, inventiveness, and power. At this stage the student already has a greater measure of these attributes than he would have had at Grade 10. It is time to extend them in an area central and immediately applicable to the main stream of college mathematics. Several suggestions for the content of such a course are:

(1) Point Set Topology.

A course in the elementary notions of point set topology with particular emphasis on plane sets of points. Use the Euclidean metric to define an open set and then proceed with a development of topological notions. This sort of a course has been written for Belgian eleventh graders:

Papy, Mathématique Modern 8 (Arlon 8)
Premieres leçons d'analyse Mathématiques

The conduct of the course is intended to be highly constructive and to be motivated by many examples. The elementary theorem do not require sophisticated arguments nor esoteric counterexamples for understanding.

Topics: Distance function; properties of a metric.
Open and closed sets; neighborhoods.
Notion of a topology; base.
Adherent points; closure; interior; frontier.
Connectivity; subspaces, induced topology.
Product spaces.
Homeomorphism; function.
Continuity.

Graph Theory.

Note: Many of these topics are discussed in Chinn and Steenrod, First Concepts of Topology, New Mathematical Library.

(2) A Course in "Systems". (Working with several representative systems.)

1. Boolean Algebras.

   1-1. Algebra of subsets of a set, algebra of propositions, algebra of circuits.

   1-2. Boolean algebra; partially ordered sets, search for additional Boolean algebras; subalgebras, do the properties characterize set algebras? Representations of finite algebras.

2. Groups.

   2-1. The group of plane isometries, the translation and rotation subgroups, the orthogonal subgroup; patterns with translation symmetry, possible patterns of lines of symmetry; determining symmetry groups of regular polygons and polyhedra.

   2-2. Permutations groups; substitution ciphers, decoding, code of a code; orbits, cycles, even and odd permutations, finding symmetric polynomials.

   2-3. Groups; subgroups, index 2 subgroups; search for new groups, finding I^+; cyclic groups, groups of orders 2 - 6; isomorphism, representation as permutation groups; do subgroups form a Boolean algebra?

3. Geometric Systems.

   3-1. The parallel postulate in Euclidean geometry, consequences and equivalent properties; properties of spherical projection (as hemispherical) geometries. (See example 4-1 below for continuation.)

4. Further Investigations.

   4-1. (a) Each city is on at least one county route.

   (b) Every pair of cities is connected by exactly one county route.

(c) Every county route has exactly 3 cities on it. What does this imply about the configuration of routes and cities?

4-2. What properties do length, area, volume have in common? What do these properties imply? Where else do these properties turn up?

4-3. The squaring function S has the property $S(x + y) - S(x) - S(y) = 2xy$. Are there any other functions with this property? If yes, try to characterize S with additional properties.

4-4. Similar to above with $L(x + y) = L(x) + L(y)$ and $F(x + y) = F(x) + F(y)$.

And so on ...

(3) Number Theory.

At Grade 11 we should be able to offer a substantial one-semester course in elementary number theory. This course could proceed along the lines of SMSG, Essays in Number Theory. This course would not list axioms for the integers, but rather would discuss and assume the crucial properties of division, Archimedean ordering, definition of prime number, and unique factorization. The course would stress the nature of proof rather than axiomatics. We concede that number theoretic proofs are much closer to "tricks" than to general methods. This places a greater premium on the student's inventiveness than on his ability to make minor alterations in a general method of proof.

From Grade 7 we have the notion of g.c.d. and the Euclidean algorithm. But we do not have much practice with linear Diophantine equations.

Topics:
- Tests for primes and divisibility.
- Infinitude of primes and of primes of the form $4k - 1$.
- Conjectures on primes.
- Congruences; Chinese remainder theorem.
- Fermat's little theorem; Wilson's theorem.
- Ring properties of the integers modulo n.
- Pythagorean triples.
- Farey Series.
- Fibonacci Series.

Gaussian integers; unique factorization theorem.

$\Sigma \frac{1}{p}$ diverges.

Four squares theorem.

(4) And so on ...

Comment on The One Unit Axiomatic/Deductive Block

The preceding document is a revised summary of the group discussions of a possible one-semester course in "Synthetic/Deductive Block". Of the possibilities listed some would prefer in order:

(1) Type A - A Geometry Block (Leading to some "geometric power").

(2) Type B (2), 3 - Geometric Systems.

(3) Type B (2), 2-3 - Groups; subgroups, ...

Some expressed the feeling that the other choices seemed too specialized, and not sufficiently rich in "real life" problem situations.

Here is the other version, followed by some comments. (This is a first draft, of an overly inclusive listing of topics and comments.)

(Synthetic) Geometry of the Plane
("Synthetic" is not a good word here)

1. Introduction.
   (1) Purposes of the course:
      (a) Classical -- It is part of the world's intellectual history.
      (b) Interesting -- Many of our present mathematicians first became fascinated with the "originals" in the traditional geometry courses.
      (c) Psychologically satisfying -- "I solved it." There is a clear sense of "closure" -- you know when you have a solution or you have not.
      (d) Esthetically gratifying. "There is a beautiful proof for that." "Euclid alone has looked on beauty bare."
      (e) It is perhaps the best example of an extensive deductive system with many non-trivial proofs and conclusions. It is a model for such deductive systems.

2. The Ground Rules.
   - (1) Clearly stated:
     - (a) Axioms -- intuitively acceptable; (? should we include axioms on order and separation?) no emphasis on minimal set; use with "postulated" theorems, definitions, etc., in deductive schemata.
     - (b) Equivalent axiom sets. (categoricity?)
       Euclid V and Playfair.
     - (c) Alternate axioms lead to alternate geometries. History of beginning of non-euclidean geometry; finite (miniature) geometries.
     - (d) Allowable construction tools: compasses, unmarked straightedge. Possible constructions depend on allowable tools. (relation to "constructable" numbers?)
     - (e) Restricted construction tools: compasses alone: Mascheroni; ruler alone: incidence geometry, projective; ruler and one circle.
     - (f) Nature of proof. Indirect proof, contrapositive, converse.

3. Straight Line Figures.
   - (1) Incidence, Congruence:
     - (a) Line. Segment, ray, etc. (order, separation?)
       What is length? Measures of length.
     - (b) Angle, measure, classification (interior?) (sensed?)
     - (c) Pairs of angles, sum, difference, vertical, adjacent, complementary, supplementary.
     - (d) Parallel and perpendicular pairs of lines, "distance" from point to line; transversals, and related angle pairs.
     - (e) Triangles. Classify by sides, angles; sum of angles = 180 .
       Congruence; Datum: when is a triangle determined.
       Constructability: construct (a) triangle given.

       (Through the whole course the matters of a datum, and constructability, will often appear. These are sources of many non-trivial problems, e.g., the construction of a figure with constraints; e.g., construct an equilateral triangle on three given parallel lines; describe a square in a given triangle.) (degrees of freedom?)

(f) Quadrilaterals, general then special, parallelogram family tree.

(g) Polygons. (convex? crossed?) Angle sums, regular, (star?)

4. Circles.

(1) Terminology:

(a) "Length of arc, circumference." What is it? (rectifiability?)

(b) Two measures of arc; length, turning.

(c) Circles and lines: diameter, radius, chord, tangent, secant, angle. (tangency?)

(d) Measures of related angles and arcs.

(e) Construct a circle under given constraints: e.g., with triangle; inscribed, circumscribed, escribed; problems of Appolonius.

5. Ratios and Proportionality.

(1) Commensurability?

(I think it would be good to develop the non-numerical aspects of these ideas.)

(a) Similar (homothetic) figures.

(b) Pythagorean theorem and application.

(c) "Product" theorems for segments: intersecting chords; tangent; secant; Theorems of Ptolemy, Ceva, Menelaus.

(d) Constructions, e.g., fourth proportional with compasses only.

6. Area.

(1) What is it? How is it measured? (Which geometric figures have area?)

(a) Find the area: square, rectangle, parallelogram, triangle, polygon.

(b) Circular regions; and combinations with others.

(c) Compare the areas: rectangles with equal bases; triangles with equal bases.

(d) Compare the areas of similar figures.

(e) Constructions: e.g., square equivalent to rectangle, triangle, etc. (Impossible: square equivalent to circle); triangle equivalent to sum of two triangles, etc.

7. Miscellaneous.

(1) (Since the outline is too long already, it might as well be way too long.)

(a) Topology: order and separation, partitions, connectedness.

(b) Convexity, boundedness.

(c) Continuity (nested intervals, regions).

(d) Locus problems (perhaps in earlier sections).

(e) A much more extensive treatment of geometric inequalities than has been traditional on this level.

(f) More dynamism in the geometry of this level; a changing rather than a static relationship. (What happens to the area of $\triangle ABC$ if we keep the lengths of $\overline{AB}$ and $\overline{AC}$ fixed, but increase $\angle A$?)

Comment on (Synthetic) Geometry of the Plane

In the early discussions, the group considered a one-semester course in synthetic geometry, built on the rather extensive geometric background already covered through Grade 9. This outline was intended to be a basis for more specific discussions of such a course. Much of it now appears in various forms in other papers, but here are a few points which don't:

1. There probably should be more inequalities than have usually been treated in synthetic geometry.
2. We may need a more dynamic approach (what happens if we increase this angle, segment,a little?) rather than the traditional static one.
3. Some think construction problems should be much more extensively used. (Constraining conditions, special tools, proofs of possibility and impossibility, multiple solutions, and so on.)
4. There should be a free flow between two-and three-dimensional situations with specific efforts to develop good spatial imagination:

    (1) Some solid geometry problems are really plane geometry problems in different planes.

    (2) Some solid geometry problems are natural extensions from plane geometry.

5. The emphasis should not be on teaching a lot of geometry, but on how to do geometry. Rather than "Prove this theorem", use "If, in this situation we know . . . , then what else can we figure out?" Try to discover first, then prove.

Outline for Year of Vector Geometry, Linear Algebra, and Elementary Functions, Level 10

Comments:

Vectors will be used whenever helpful and appropriate in the study of functions and their graphs. Sometimes the function to be studied may be introduced by considering motion along a given path; i.e., motion with constant velocity along a line leads to the linear function. Such considerations lead naturally to the study of general vector functions. Vector functions may be mappings of points on the real line or portions thereof onto points on another real line or points in the plane or even onto points in three-space. Similarly we may want to map points in the plane into other points in the plane or into 3-space. The study of analytic geometry arises naturally in being specific about locating images of points under such mappings and specifying the set of points constituting the range of such functions. In order to reinforce and extend the kind of manipulative facility which students need, translation from the parametric equations which come out of consideration of vector functions into the corresponding rectangular equations should be continually stressed.

If we are to use vectors and vector functions in this way, in the initial chapter the transition from vectors as displacements to vectors associated with the displacement of the origin to a point should be made as in Paige's Chapter 1, Section 1-2.

The first half of the course then will do much analytic geometry and elementary functions and their graphs using vectors and ideas of motion whenever helpful and appropriate.

The second half will use vectors in the setting of linear algebra. The study of geometric properties of vectors (parallel and perpendicular vectors - the dot product) will be used to derive the equations for planes and for lines in space. (Proof of geometric theorems by vector methods should come in somewhere.) Matrices will be introduced as a convenient way of displaying information and then it will be shown that they can also be used to denote vectors and transformations (reflections, rotations, contractions, stretching, etc.). Just enough matrix algebra will be introduced to talk about transformations and to use elementary row operations on matrices to solve systems of equations. Linear dependence may be discussed in connection with the nature of the solution of systems of equations. Rotations may be used to simplify quadratic forms (conics).

OUTLINE FOR YEAR OF VECTOR GEOMETRY, LINEAR ALGEBRA, AND ELEMENTARY FUNCTIONS, LEVEL 10

Part 1

Functions and Their Graphs

Chapter 1. Vectors. (See Paige, Sample Chapter 1)

1-1. Introduction.

1-2. Addition of Vectors.

1-3. Scalar Multiplication.

1-4. Space Coordinates.

Chapter 2. Straight Lines and Linear Functions.

2-1. Motion along a Straight Line.

$\vec{F}(t) = \vec{A} + t\,\vec{B}$.

2-2. Ways of specifying position of point which moves along a straight line with constant velocity.

$\vec{F}(t) = [x(t), y(t)] = [a_1 + tb_1, a_2 + tb_2]$.

Pull out parametric equations for path of point. Examine what happens when $\vec{B} = [1,0]$; $\vec{B} = [0,1]$; arbitrary $\vec{B} = [b_1,b_2]$.

2-3. Slope

$$m = b_2/b_1 \; ; \quad \text{get} \quad \frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} .$$

Look at special cases for $\vec{A} = [a,0]$ and $\vec{A} = [0,b]$.

2-4. Do the same for path along line in 3-space

$\vec{F}(t) = [x(t), y(t), z(t)] = [a_1 + tb_1, a_2 + tb_2, a_3 + tb_3]$.

Get parametric equations for line in space.

Chapter 3. General Vector Functions.

3-1. Vector functions as mappings.
Maps of one line onto another; interval onto point set in the plane; one plane onto another; etc.

3-2. Coordinate functions.
Look at a sequence of increasingly difficult maps; e.g.,
$\vec{F}(t) = [t,t]$; $\vec{F}(t) = [t,t^2]$; $\vec{F}(t) = [t,t^3]$;
$\vec{F}(t) = [t,t,t]$; $\vec{F}(t) = [t,t,t^2]$; etc.

3-3. Parametric and rectangular equations for curves.
Pull out explicit sets of parametric equations $x = t$, $y = t$; $x = t$, $y = t^2$; $x = t$, $y = t^3$; and their rectangular counterparts $x = y$; $y = x^2$; $y = x^3$; etc., and draw their graphs.

3-4. Parameters having geometric or physical significance.
Hypocycloid, Ellipse, Circle, etc. Get parametric equations, find corresponding rectangular equations and discuss advantages of each.

Chapter 4. Polynomials and Rational Functions.

4-1. Parabolic paths of moving bodies (comets, bullets, etc.).

4-2. Get quadratic equations from converting parametric equations to rectangular form.

4-3. Other problems which lead to the consideration of quadratic equations and their roots.

4-4. Relations between roots and coefficients.

4-5. Quadratic inequalities and their graphs.

4-6. Generalize to polynomials of higher degree. (Find problems which lead to cubics and quartics -- volumes of boxes and cylinders with minimal surface area for fixed volume, etc.).

4-7. Theorems necessary to find roots of polynomials and other information helpful in drawing a rough graph.
Factor Theorem; Remainder Theorem; Location Theorem; Rational Roots Theorem.

4-8. Approximating real roots; flow chart for linear approximation.

Chapter 5. Trigonometric Functions.

5-1. Simple harmonic motion.
Define Cosine and Sine as coordinate functions of vector function which describe the path of a point which moves with constant velocity (counterclockwise) around the unit circle. Connect with previous version of trigonometry. Review radian measure and connect with parameter describing the original vector function.

5-2. Describe tangent function as coordinate function of point which moves along vertical line tangent to unit circle;
$\vec{T}(t) = [1 , \tan t]$ maps $(-\frac{\pi}{2} , \frac{\pi}{2})$ onto the line $x = 1$.

5-3. Use definition of functions to obtain relations between functions of general angles and functions of acute angles.

5-4. Fundamental identities and addition formula.

5-5. Derive law of cosines.

5-6. Solve some trigonometric equations.

5-7. Solution of some triangles (simple computations not with logs).

Chapter 6. Polar Coordinates and Rectangular Coordinates.

6-1. Polar coordinates and rectangular coordinates.

6-2. Polar coordinates and vectors.

6-3. General conic in polar coordinates.

6-4. The circle and the ellipse.

6-5. The hyperbola.

6-6. Rectangular equations of conics.

Chapter 7. Vectors and Complex Numbers.

7-1. Points in the plane.
Point out the relation between rectangular coordinates of a point in the plane, polar coordinates of a point in the plane, components of vector considered as the displacement of the origin to the point in the plane. Then explain the representation of complex numbers as points in the complex plane. Point out the connections between polar coordinates of the point, polar form of complex number, rectangular

coordinates of the point and the a + bi form of the complex number.

7-2. Problems with addition and multiplication of complex numbers.

7-3. Multiplication of complex numbers in polar form.

7-4. DeMoivre's Theorem and nth roots of complex numbers.

Chapter 8. Exponents and Logarithms.

8-1. Review of properties of exponents and logarithms. (Finger exercises)

8-2. Rough graphs of these functions.

8-3. Use of the exponential function to describe growth and decay.

8-4. Computations with logarithms.

8-5. Use of logarithms to solve triangles.

Part 2

Vectors and Linear Algebra

Chapter 9. Vector Algebra.

9-1. Review of properties of addition and scalar multiplication.

9-2. Inner product.

9-3. Conditions that vectors be parallel or perpendicular.

9-4. Vector proofs of geometric theorems.

Chapter 10. Lines and Planes in Space.

10-1. Review of coordinates and vectors in 3-space.

10-2. Vector equation of a plane.

10-3. Rectangular equations for plane.

10-4. Vector equation for line in space.

10-5. Parametric equations for line in space.

10-6. Intersections of lines and planes.

10-7. Other geometric problems with lines and planes.

Chapter 11. Matrices.

- 11-1. Introduction.
  Matrices as ways of systematically recording data. Motivation.
  (See *Introduction to Matrix Algebra*, SMSG)
- 11-2. Order of a matrix; addition of matrices.
- 11-3. Multiplication of a matrix by a scalar.
- 11-4. Multiplication of matrices.
- 11-5. Some properties of matrix multiplication.

Chapter 12. Matrices and Vectors.

- 12-1. Representation of vectors as column or row matrices.
- 12-2. Geometric interpretation of multiplication of column matrix (vector) by scalar; sum of two column matrices (vectors).
- 12-3. Vector spaces and subspaces.
  Line as subspace of the plane; plane as subspace of 3-space.
- 12-4. Transformations of the plane expressed in matrix notation.
  (See Chapter 5, *Introduction to Matrix Algebra*, SMSG)
- 12-5. Linear transformations.
- 12-6. Characteristic Vectors - Invariants under linear transformations.
- 12-7. Rotations and Reflections.
- 12-8. Rotations to simplify equations of conics.

Chapter 13. Matrices and Solutions of Systems of Equations.

- 13-1. Representing systems of equations by matrices.
- 13-2. Equivalent systems of equations and elementary row operations on matrices.
- 13-3. Diagonalization method for solving systems of equations.
- 13-4. Inverse of a matrix.
- 13-5. Linear dependence and singular matrices.
- 13-6. Analysis of solutions of systems of equations.
- 13-7. Application to linear programming problems.

A Tenth Level Course in Analytic Geometry and Algebra

The outlining group for Grades 7, 8, and 9 of 1966 precede their discussion of the 9th Grade material with the remark, "We felt that the students' preparation was now adequate for the presentation of substantial mathematical ideas. Hence, our outline became brief, and the arrangement was tentative with considerable freedom left to the writers." The proliferation of possible alternatives for the mathematics of Grade 10 obviously indicates that many of us believe the student will be sufficiently prepared after Grades 7, 8, and 9 to follow a consistent mathematical development. It would be regrettable if we did not pursue this possibility.

I propose that an integrated course in "Analytic Geometry and Linear Algebra" be devised which presents to the student a modest introduction to present day emphasis in mathematics at the college level. It would be appropriate for me to state briefly what I feel a substantial portion of this emphasis to be:

(1) "Coordinate free" formulation of definitions whenever possible in algebra and generalization wherever reasonable in analysis. Thus, we find that determinants are now multilinear functions from ordered sets of n vectors to a field F rather than a notation for solving linear equations with an attendant exercise in line drawing for the special cases $n = 2$, $n = 3$. Similarly, Green's Theorem no longer appears in an advanced calculus text as a detailed exercise in evaluating the line integral

$$\oint_c (Pdx + Qdy) ,$$

but as a special case of exterior algebra and differential forms.

(2) In following the trends of (1), a considerable portion of material centers around a set of operators, (often some algebraic system), and a space upon which these operators act. Questions which often arise are:

(a) Given the set of operators, what are "invariants" in the space which in some way characterize the operators? Or, from another point of view, what are "invariants" of the spaces which might characterize spaces that may be mapped upon each other by an operator?

I am convinced that we should begin, in any study of analysis in the 10th level, to lay the experimental foundations for these future possibilities. Certainly, the concepts of vector spaces, groups and invariants will arise and I see no reason why a modest beginning cannot be made through a course in "Analytic Geometry and Linear Algebra".

I propose an integrated course in "Analytic Geometry and Linear Algebra" in which the following themes are to be guidelines for the writers:

(1) I would hope that an analysis of the Euclidean group of motions acting upon a plane and in space would be a unifying concept throughout the early part of the text with alternative interpretations. This analysis would begin with translations (call them displacements, or whatever you may wish) and proceed to reflections, and rotations with dilations interspersed to provide the geometrical background for a vector space. Later the affine transformations would give ample opportunity for vector and matrix applications. I do not believe that the projective group could be introduced to give a "geometric invariant" introduction to conics. It might be nice to try.

(2) A continuing development of the function concept with a corresponding development of the algebra of functions to give illustrations of vector spaces, algebra of polynomials, etc. I see that both linear and non-linear situations will arise with ample opportunity to motivate and use matrix notation without emphasis on matrix notation as a bookkeeping device.

(3) A conscious effort to motivate and present "coordinate free" definitions should be done when possible. Then formulate the same concept in coordinate systems. This will illustrate that an explicit coordinate formulation will depend upon the coordinate system chosen. Many opportunities will arise to apply matrix notation to "changes of coordinates" and the interpretation of these changes as operators acting on the plane and in space. I would certainly prefer that parabolas be defined as the locus of points equidistant from a point

and a line rather than the set of all points satisfying

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 ,$$

where $B^2 - 4AC = 0$. (I must admit that it would be difficult to justify enough projective geometry to motivate conics as projective invariants.)

The use of coordinate systems to investigate geometric concepts will force one to play "coordinate free" interpretations against "coordinate system" formulations whenever possible.

(4) Vector notation should be used extensively, even where it is little more than a bookkeeping device.

(5) Motion along lines and curves could easily be used as a motivating device or as an application of the mathematics developed. I see no reason to stress the former alternative or the latter.

A first approximation to the course outline proceeds as follows and I repeat for emphasis (louder, if possible) that I do not believe that there should be any sharp distinction between analytic geometry, linear algebra (viewed as vectors and matrices) in the sequence.

Chapter 1. Vectors.

1-1. Introduction.
(Vectors as representation of translations.)

1-2. Addition of vectors.

1-3. Dilations.
Geometric viewpoint as operator acting on space. Motion along line.

1-4. Geometric Applications of Vectors.

1-5. Space Coordinates.

Chapter 2. Inner Product.

2-1. Vector projections and definition of inner product as
$\vec{A}\cdot\vec{B} = |A||B| \cos \angle (A,B)$. (Note that this is "coordinate free".)
Now look at coordinate interpretation of inner product.

2-2. Vector components in terms of $\underset{\sim}{i}$, $\underset{\sim}{j}$ (and $\underset{\sim}{k}$).

2-3. Review slope, equations of lines in analytic and vector form.

2-4. Projection of linear constant motion along coordinate axes.

2-5. Constant motion on circle.

2-6. Simple Harmonic Motion.

2-7. Law of Cosines.

2-8. Review of trigonometry including, if advisable, triangle solutions.

Chapter 3. Rotations in the Plane.

3-1. Geometric View.
(Heuristic Invariants)

3-2. Coordinate Representation of Rotations.
(Here matrix multiplication arises naturally)

3-3. Rotations as operations on points.
(Again an operator viewpoint. One could also look at the rotation as operating on vectors as a first glimpse at linear transformations.)

3-4. Use of matrix notation to obtain trigonometric formulas for $\sin(\alpha+\beta)$, $\cos(\alpha+\beta)$ -- by considering composition of rotations -- from matrix-geometric viewpoint.

3-5. Isometries of the plane.
(Here combine rotations, translations, reflections and consider coordinate interpretations.)

Chapter 4. Geometry of Space.

4-1. Translations again.
(Heuristic Invariants)

4-2. Constant motion along line.
(Vector valued view. Description by various coordinate systems under translation.)

4-3. Motion along curves in space.

4-4. Rotations in space.
(View as operator on points. Also as operator on vectors.)

At this point I would break the treatment and look at polynomials with the motivation starting from motion along curves in space arising in Chapter 4.

Chapter 5. Polynomials.

(See polynomials introduced as functions but note the slight emphasis on vector space of polynomials, algebra of polynomials, isomorphism.)

Chapter 6. Rational Functions.

Chapter 7. Return to Geometry with a Look at Affine Transformations.

Chapter 8. Affine Invariants.

Chapter 9. Matrix Applications.

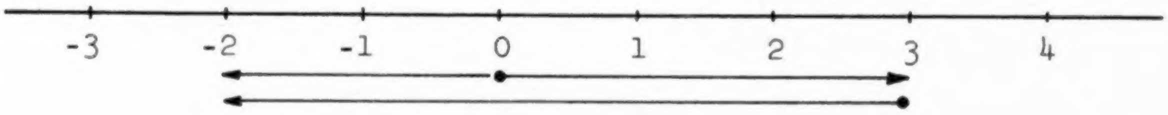
The following is an illustration of how vectors could be introduced. (Vectors as representing translations.)

Sample Chapter 1

Vectors

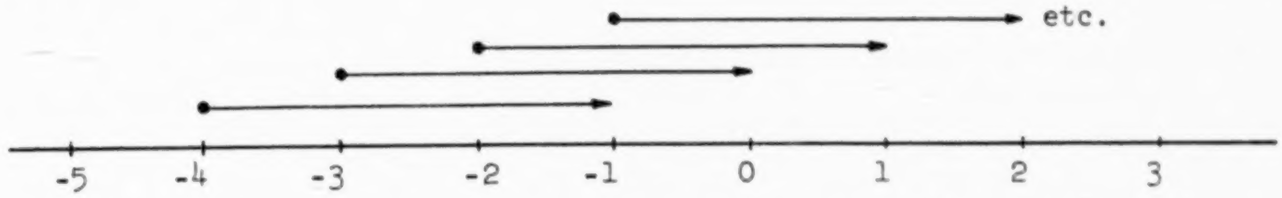
Section 1-1. Introduction.

We have observed many times that a coordinate system on a line, in a plane or in space permits us to describe quite accurately physical situations or mathematical operations. For example, the number line permits us to describe quite geometrically the addition of integers -- by means of directed distances along a line



Geometrical description of $3 - 5 = 2$.

Similarly, we could picture the statements x + 3 for all integers x by



Geometrically, we are mapping the points of the number line corresponding to the integers onto the points of the number line. We associate with a point corresponding to x the point corresponding to x + 3 . Looked at from a slightly different point of view, we have a function of the integers

$$f : x \to x + 3 \, .$$

[At this stage, one could pursue some more function concepts, "time to points on a line" for a moving object a la Lister if one wishes to. I feel that this is somewhat a matter of taste and I would go on.]

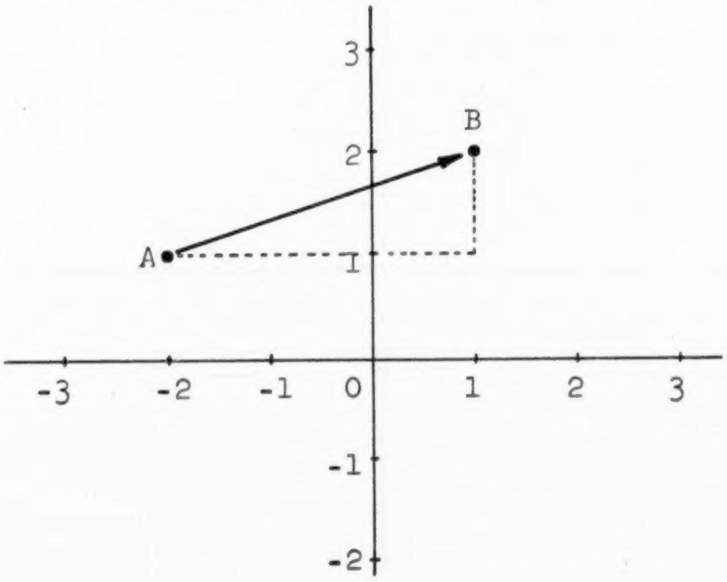
Question: Are there any geometrical invariants under this mapping of the line? (Student participation.)

What is the geometrical interpretation of the function $f: x \to 3x$ along a line? (A change of coordinates question?)

[This line might be deadly.]

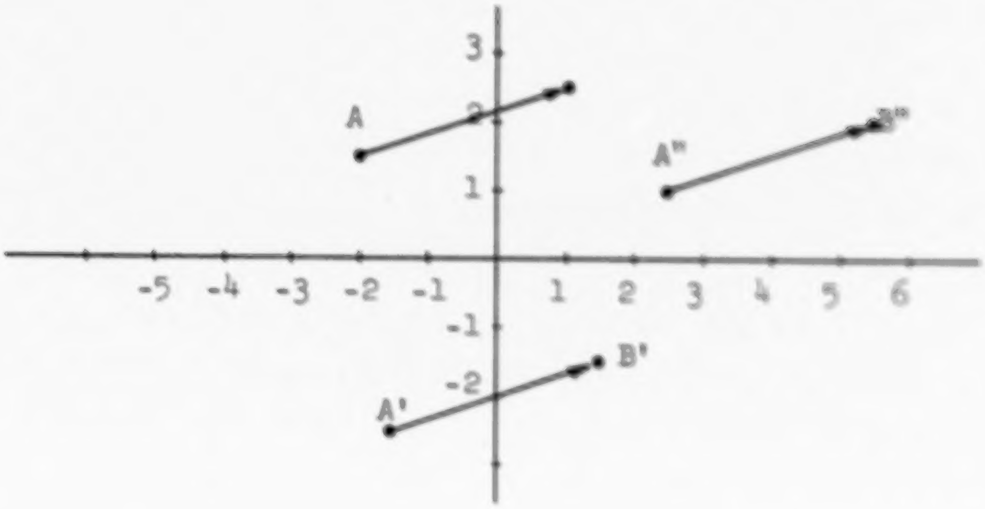
The questions we have asked about the mapping $f: x \to x + 3$ for points on a line certainly raise similar questions about the mappings of points in the plane.

Let us begin with the simplest case. We establish a coordinate system in a plane π



and consider a mapping of π which maps a point A upon a point B three units east and one unit north.

We can plot the images of many points



and, in terms of coordinates, we can write for any point

$$(x,y) \rightarrow (x + 3, y + 1) .$$

It should be clear that similar to the line, one vector from a point to its image represents what happens to all points; namely 3 east, 1 north.

What geometrical properties do not change under this mapping?

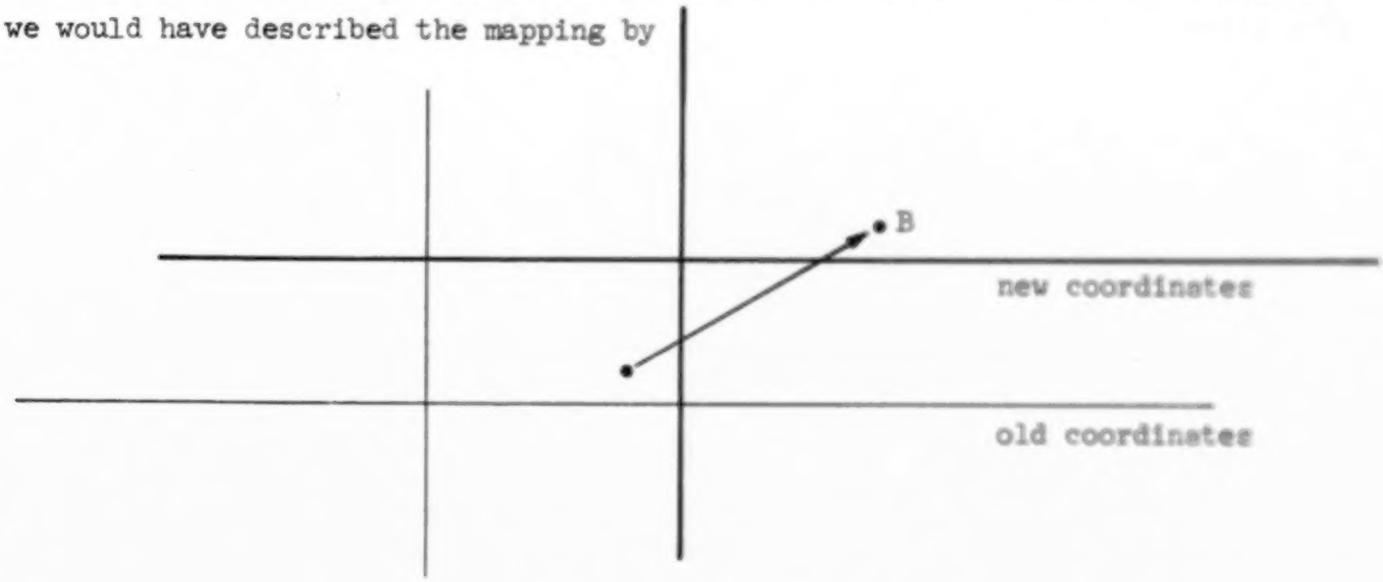
(a) Distance?

(b) Angles between lines?

(c) Do triangles map into triangles?

(d) Etc. (Student participation.)

Now let us note that if we had chosen another parallel coordinate system we would have described the mapping by



A goes into a point 3 east, 1 north and the mapping would give rise to the function

$$f: (x', y') \rightarrow (x' + 3, y' + 1) .$$

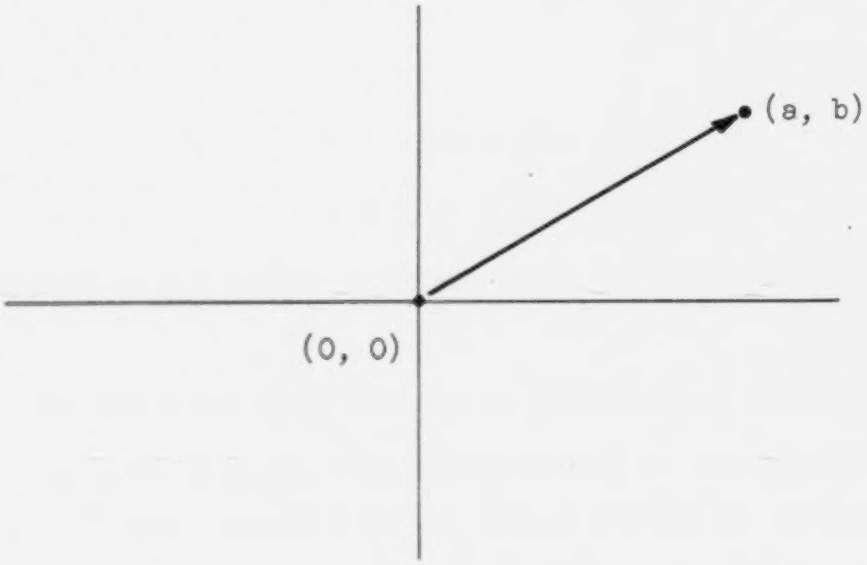
Again one vector from a point to its image represents the "movement" of all points.

Section 1-2. Addition of Vectors.

Any mapping of the plane given by a function of the form

$$f: (x,y) \rightarrow (x + a, y + b)$$

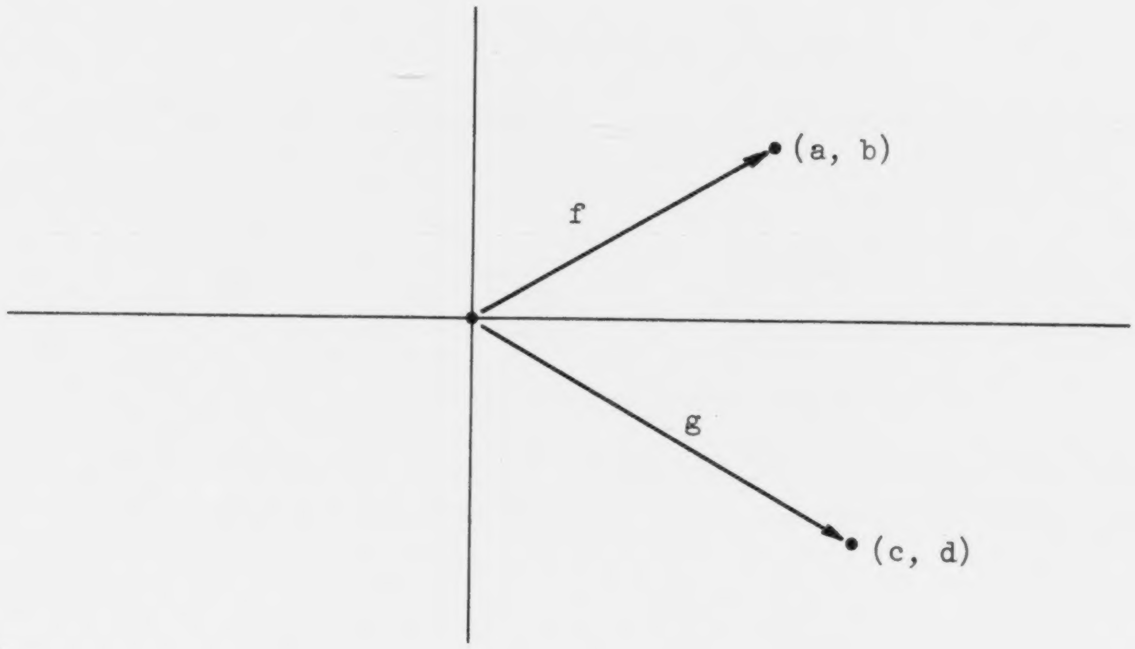
can be represented by a vector indicating the image of the point (0,0) . Thus with each point in the plane (a,b) we may associate a vector from (0,0)



and this vector can be used to describe the translation (this should have entered earlier)

$$f: (x,y) \rightarrow (x + a, y + b) .$$

Let us assume that we have two translations f and g which are represented by the vectors to the points (a,b) and (c,d) .



Thus

$$f: (x,y) \rightarrow (x + a, y + b)$$

$$g: (x,y) \rightarrow (x + c, y + d) .$$

What happens to a point A if we first apply f then g ? (Student participation.)

[We are now led naturally to composition of functions and addition of vectors.]

[We are primarily interested in the students developing a feeling for addition of vectors with various interpretations: Force Diagrams; etc.]

Section 1-3.

Now develop a geometrical background and feeling for the mappings of the plane given by

$$f: (x,y) \rightarrow (tx, ty)$$

leading to a natural definition of scalar multiplication of vectors.

We can now give a description of motion along a line (any line) in the plane with scalar multiplication and addition of vectors at hand. (Use only constant velocity along line.)

Section 1-4. Geometric Applications of Vectors.

(Linear independence leading to review of solutions of equations.)

Section 1-5. Space Coordinates.

Here we set up a coordinate system in space and indicate that the question of translations in space lead naturally to the definition of vectors in space and we can ask questions similar to those of the preceding sections as to invariance of various geometrical concepts and the definition of vectors in space with addition and scalar multiplication defined.

Again constant motion along line could be investigated.

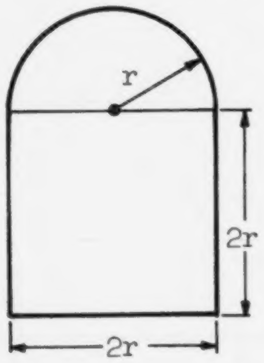
[I believe this section should be an introduction and that we should explain that in general we shall stick to the plane.]

Sample Chapter 5

Polynomials

Section 1.

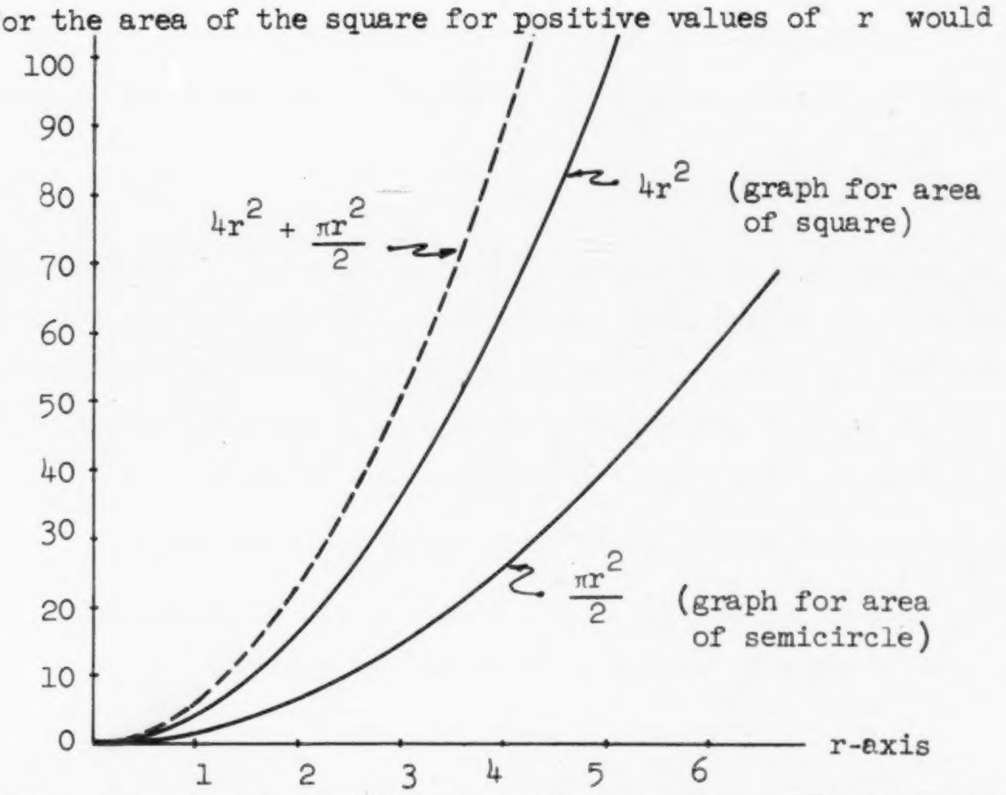
Many problems in mathematics require the addition of functions. For example, if you are asked to compute the area of the following figure



you can first compute the area of the square and then add the area of the semicircle:

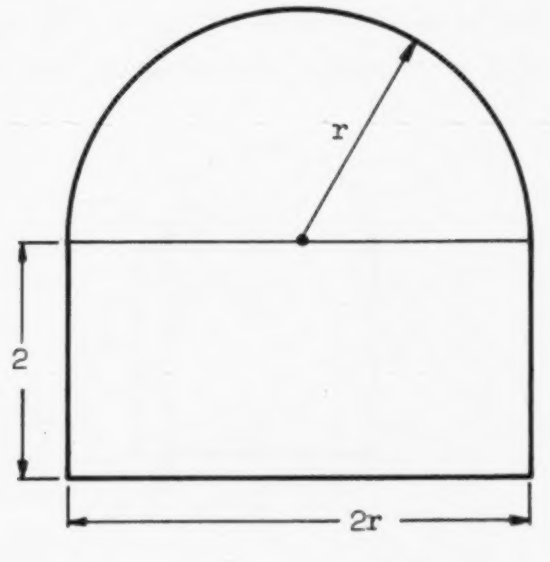
$$\text{Total Area} = 4r^2 + \frac{\pi}{2}r^2 \ .$$

A graph for the area of the square for positive values of r would look like



We could graph the area of the semicircle on the same axes. Now, in order to graph the total area of the figure, we would merely need to add together the heights of the graphs for each point r .

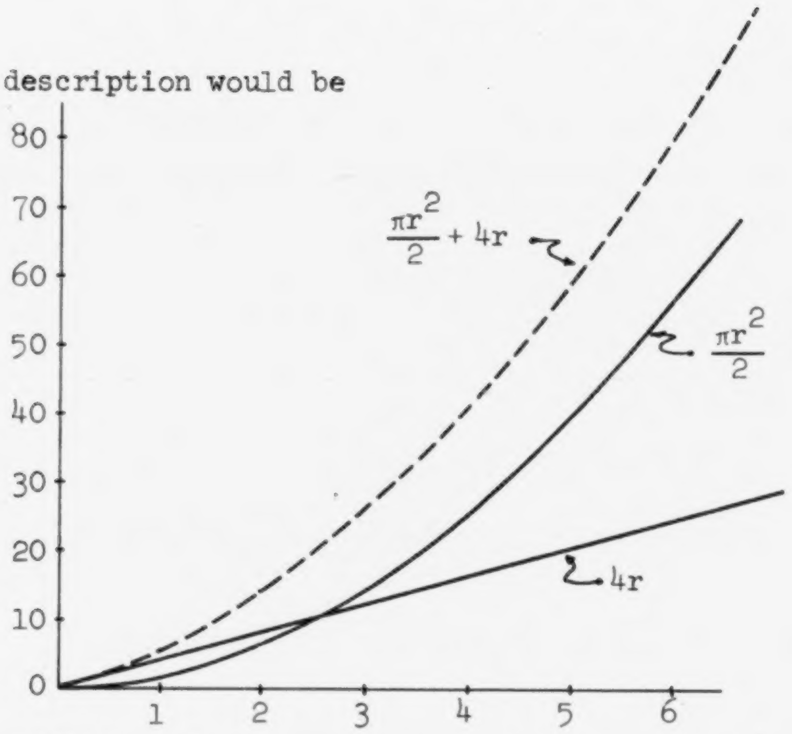
A similar situation would prevail if we were asked to compute the total area of the figure



Here, the total area is given by

$$\text{Total Area} = \frac{\pi}{2}r^2 + 4r \text{ ,}$$

and our graphical description would be



The pointwise addition of our graphs suggests a useful definition of the sum of two functions. Hence, let

$$f: x \to f(x)$$
$$g: x \to g(x)$$

be two functions from the rationals to the rationals and define

$$f + g : x \to f(x) + g(x) .$$

Note No. 1.

At this stage, there should be no reservation in considering many examples using

$$f: x \to x^3 + x - 1$$
$$g: x \to x^5 - x + 3 ,$$

where (polynomials) of any degree are used.

Note No. 2.

After considerable plotting of addition, the function $af: x \to af(x)$ should be motivated and examples given.

It should then be pointed out that our addition and scalar multiplication of functions satisfy the same properties as vectors; i.e., we have a "vector space" of functions.

Section 2. Multiplication of Functions.

You have used the pointwise addition of functions as we have described them for sometime! Look at the addition of integers. For example, if you add

$$\begin{array}{r} 7{,}831 \\ \text{and}\quad \underline{347}\ , \end{array}$$

isn't this precisely

$$\begin{array}{lllll} 7 \times 10^3 + & 8 \times 10^2 + & 3 \times 10 + 1 \\ & 3 \times 10^2 + & 4 \times 10 + 7 \\ \hline 7 \times 10^3 + & (8+3) \times 10^2 + & (3+4) \times 10 + (1+7) \end{array}$$

and this is merely the pointwise addition of the functions

$$f: x \rightarrow 7x^3 + 8x^2 + 3x + 1$$

and

$$g: x \rightarrow \qquad 3x^2 + 4x + 7$$

at the point $x = 10$.

Is it possible that multiplication of integers has a similar interpretation in terms of functions?

When we multiply

$$\begin{array}{r} 346 \\ \underline{24}\ , \end{array}$$

we may think of this as

$$(3\times10^2 + 4\times10 + 6) \cdot (2\times10 + 4)$$

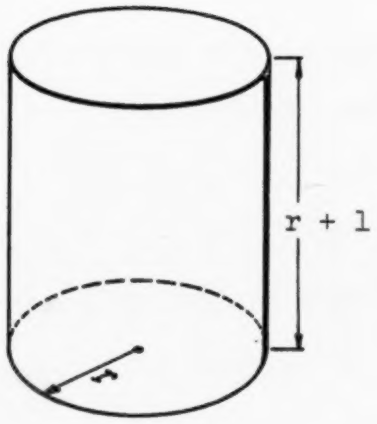
and this is the product of the functions

$$f: x \rightarrow 3x^2 + 4x + 6$$

and $g: x \rightarrow 2x + 4$

at the point $x = 10$.

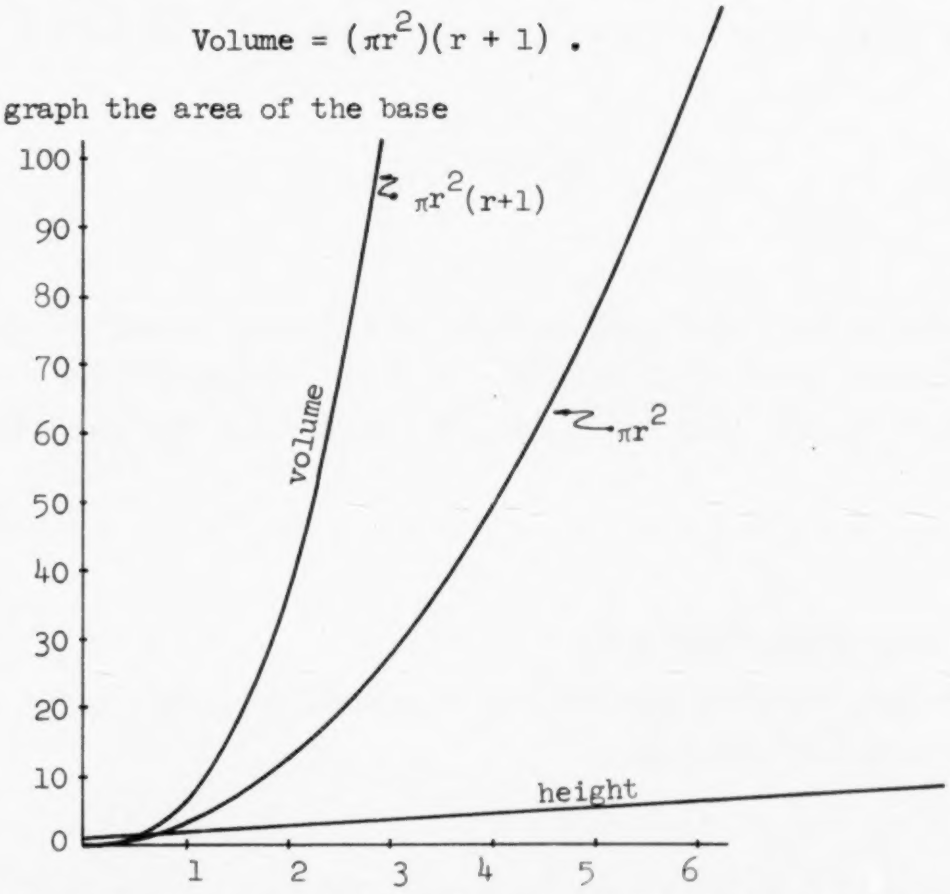
Graphically, there are times when we might wish the product of two functions at many points and not just the point $x = 10$. For example, let us look at the following problem: What is the volume of the following figure?



The base is of area πr^2 and to obtain the volume we compute

$$\text{Volume} = (\pi r^2)(r + 1) .$$

Again, we may graph the area of the base



We can obtain the volume for any value of r by multiplying the heights of the two graphs at each point.

- -

Note No. 3.

Do many examples using

(polynomials)

$f: x \to x^3 + x + 1$

$g: x \to x^2 + 7$

- -

The pointwise multiplication of our graphs suggests a useful definition for the product of two functions. Hence, let

$$f: x \to f(x)$$
$$g: x \to g(x)$$

be two functions from the rationals to the rationals and define

$$fg: x \to f(x)g(x) .$$

- -

Note No. 4.

At this stage, there should be many examples where we use (polynomials)

$$f: x \to \underline{\qquad\qquad}$$
$$g: x \to \underline{\qquad\qquad} .$$

- -

Note No. 5.

It should be mentioned that we have taken "vector space" of functions and defined a "multiplication" of vectors. It might be appropriate to point out that both distributive laws hold, etc., but this risks the emphasis of too much axiomatics.

- -

Section 3. Polynomial Functions.

We have seen that the equations of straight lines lead us to consider functions (linear) of the form

$$f: x \to ax + b .$$

The motion of a falling body leads to a function of the form

$$g: x \to ax^2 + bx + c .$$

Moreover, the multiplication and addition of integers are instances of the multiplication and addition of functions of the form

$$f: x \to a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_o ,$$

where x is usually chosen to be 10 and the a_i are nonnegative integ less than 10 .

- -

Note No. 6.

Other examples of polynomial functions should be given of "degree > 2 ". Volume problems will provide many of degree 3 .

- -

A function of the form

$$f: x \to a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_o \text{ , where}$$

a_n , a_{n-1} , ... , a_1 , a_o are rational (n an integer ≥ 0) , from R_a (rationals) to R_a (rationals) will be called a polynomial function and occasionally, for the sake of brevity, we shall refer to the expression

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_o$$

as a polynomial in x .

- -

Example: When we consider motion along a line, we usually express distance as some function

$$d: t \to 3t^2 + t \text{ ,}$$

where the letters used have been chosen to suggest distance and time. It should be observed, however, that we are still thinking of a mapping of the rationals to the rationals and

$$d: t \to 3t^2 + t$$

is a polynomial function but we would refer to $3t^2 + t$ as a polynomial in t .

- -

Note No. 7.

There should now be numerous examples of polynomial functions with expressions in many letters so that the student would be willing to consider polynomials in any letter.

- -

We may add and multiply polynomial functions since these operations have been defined for any two functions. Thus, if

$$f\colon x \to a_3x^3 + a_2x^2 + a_1x + a_o$$

$$f\colon x \to \qquad b_2x^2 + b_1x + b_o$$

$$f + g\colon x \to a_3x^3 + (a_2+b_2)x^2 + (a_1+b_1)x + (a_o+b_o)$$

and $f + g$ is another polynomial function.

- -

Note No. 8.

Now the multiplication of two polynomial functions will get a little sticky because we must be careful to multiply the functional values to show that we are doing it for an arbitrary x and again we get a polynomial function. Only detailed writing will produce the correct motivation and wording.

- -

The polynomial functions are closed under multiplication and addition. Moreover, it would be nice to see what kind of functions can be generated by using some rather simple functions to begin with.

$$f\colon x \to x$$

$$f\cdot f\colon x \to x^2$$

$$f\cdot f\cdot f\colon x \to x^3$$

$$\vdots$$

$$\underbrace{f\cdot f\cdots f}_{n \text{ times}}\colon x \to x^n$$

Now if we include addition of functions we see that all polynomial functions can be generated by using one rather simple function and multiplication of functions by the constant functions

$$g\colon x \to c\ .$$

- -

Note No. 9. (Perhaps this should be a new section.)

At this stage, I would shift the emphasis to point out how we add and multiply the polynomials to get the addition and multiplication of polynomial functions (that is, I would sneak in their first peek at an isomorphism!). Specifically,

$$R_a[x] \to R_a \ .$$

- -

Section 5. Graphing Polynomial Functions.

(The purpose here is to give practice in graphing and lead up to zeros of polynomials.)

Section 6. Zeros of Polynomial Functions and Factoring Polynomials.

Section 7.

Is the function $x \to \sin x$ a polynomial function?

(Here we would do a little approximating and possibly some computing.)

A General Topical Sequence and Coordinates, Vectors, Functions: A Level 10 Course

1. Linear Functions and Coordinate change.

1-1. Line-like properties; time, temp, pressure, force.

1-2. Coordinates on a line; origin, displacements as translations; coordinatizing displacements.

1-3. Coordinate conversion functions; scale factor and shift term.

Arrow diagrams . Inverses.

1-4. Relations between properties; e.g., time-position as in motion on a line; temp-press; force-displ. of spring. How does coordinate form of such functions change if coordinates are changed? Can simple coordinate form be achieved? Composites.

1-5. Graphs of linear functions; slopes; translation of origin, change of unit.

2. Quadratic Functions and Gravitational Motion.

2-1. Scale changes in gravitational motion on a line; arrow diagram and graph interpretation.

2-2. Maximum height and completing the square; simple coordinate form of time-position function under shift change and under unit change of coordinates.

2-3. Graphs of quadratic functions; coordinate version of symmetry. Interpretation of a , b , c in $a(x - b)^2 + c$.

2-4. The inverse problem; role of $\sqrt{\ }$ (square root), partial inverses of arbitrary quadratic functions; time at which object is at given position. Problems from other quadratic relations.

2-5. The velocity problem; behavior of average displacement vectors; velocity vectors and the velocity function; velocity functions for arbitrary linear and quadratic position functions.

3. Motion in a Plane.

3-1. Specification of motion by time-position function; coordinate functions.

3-2. Constant velocity motion; displacement vectors, average displacement, velocity vector, algebra of displacements and velocities; relation of coordinate and parametric descriptions of lines.

3-3. Gravitational motion in a plane; coordinate functions, coordinate description of path.

3-4. Simple harmonic motion; uniform circular motion in unit circle, coordinate functions and their properties including addition formulas; question of the velocity function.

4. Trigonometry - Angle and Length Problems.

Problems from geodesy and astronomy leading to derivation and use of law of cosines and law of sines. The problem of computing $\sin(x)$.

5. Linear Functions of Several Variables.

5-1. Source examples; mixtures, costs, production, etc.

5-2. The inverse image problem, $ax + by = c$, graph.

5-3. Constraints leading to $ax + by < c$, graph.

5-4. Pairs of linear equations or inequalities.

5-5. Triples.

6. Affine Coordinate Systems.

6-1. Verification of non-metric plane geometric properties with linear coordinate systems.

7. Coordinates in Space.

7-1. Coordinate planes; coordinate lines.

7-2. Distance in space; spheres.

7-3. Graphs of linear functions of 2 variables; planes.

7-4. Graph of $(x,y) \to \sqrt{x^2 + y^2}$ and $(x,y) \to x^2 + y^2$.

7-5. Plane sections of surfaces; coordinate sections, coordinate plane projections, level sets of planes, cones, spheres.

7-6. Motion in space.

8. Linear Systems.

8-1. Intersections of planes, other situations leading to linear systems in 3 variables.

8-2. Vector formulation; structure of solution set.

8-3. Matrix of system; algorithm for solution.

8-4. Translational and rotational coordinate conversion in the plane, [in space].

8-5. Matrix of coordinate conversion functions.

8-6. [Non-linear maps of R^2 into R^2 , polar coordinate.]

9. Non-linear Relationships.

9-1. Cubing functions; source examples, analysis of $x \to x^3$, $x \to x^3-a$, $x \to (x-a)^3$ for monotonicity, convexity.

9-2. General cubic; change of sign and roots, factor and remainder theorem, roots and coefficients, intervals of increase and decrease.

9-3. Computation of roots; algorithm for approximation.

9-4. Reciprocal functions; $x \to \frac{1}{x}$, $x \to \frac{1}{x^2}$; source examples, e.g., radiation intensity, pressure-volume, rate-time.

9-5. Analysis of reciprocal functions for translated forms, monotonicity, convexity, asymptotic behavior, inequalities.

9-6. Polynomials; the division algorithm.

9-7. Rational functions; partial fractions.

10. Polynomials in Two Variables.

- 10-1. Sample sections of cones with vertex at origin.
- 10-2. Curves consisting of points satisfying certain distance conditions; parabolas, ellipses, hyperbolas.
- 10-3. Translational and rotational coordinate conversion.
- 10-4. Motion on a conic.
- 10-5. Curves defined by other distance conditions.

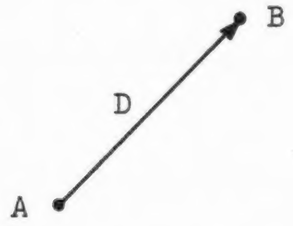
11. Geometric Transformations.

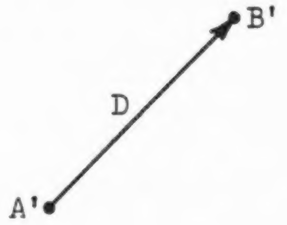
- 11-1. Motions of a plane in space; isometries.
- 11-2. Orthogonal transformations and their matrices.
- 11-3. Images of sets; orthogonal transformations with certain image specifications.
- 11-4. [Classification of orthogonal transformations and isometries.]
- 11-5. Inverses and composites; finding images and inverse images of specified sets; e.g., conics and graphs of functions.
- 11-6. Symmetries of a set; structure of certain groups of symmetries.
- 11-7. [Similarities; relation to isometries, matrices, determination of images.]
- 11-8. [Projections in plane and space.]

Displacements, Velocities, Vectors

Several introductory developments of vectors are no doubt feasible. One treatment seems to me particularly suitable in the context of motion and readily adapted to other settings. Since it is not standard a brief sketch may be in order.

Displacements are relative changes of position. Thus B is 4 miles northeast of A defines the displacement 4 miles northeast. In general displacements can be specified with arrows. D is the displacement which

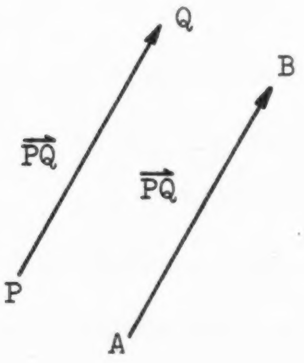


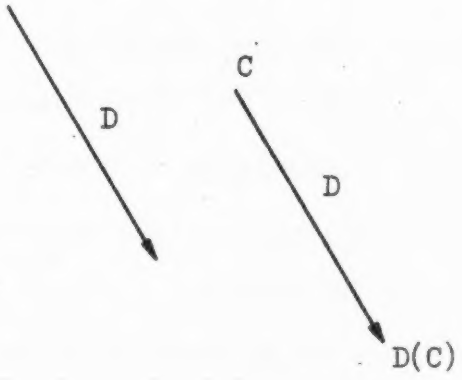


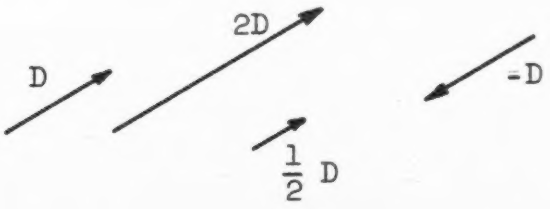
carries A into B . The ~~same~~ displacement carries A' into B' . In coordinate terms the displacement from (0,0) to (2,1) is the same as that which carries (4,3) to (6,4) .

If a displacement D and a point P are given let D(P) denote the effect of D on P .

If P and Q are points there is a unique displacement D for which D(P) = Q . Call this $\overrightarrow{PQ}$.







It is convenient to denote D(P) by P + D or D + P . It may also be convenient to denote $\overrightarrow{PQ}$ by Q - P . In any case we define

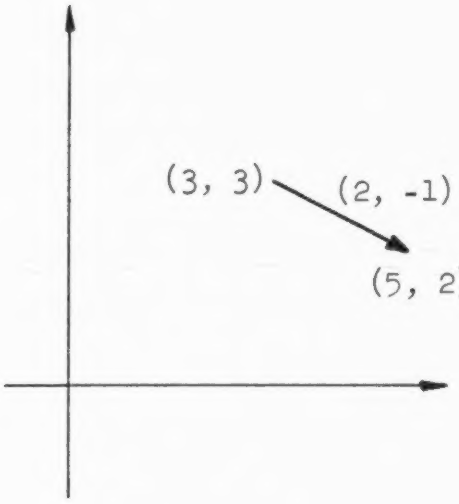
$$D_1 + D_2 : P \rightarrow D_1(D_2(P)) \quad (\text{or } D_2(D_1(P)))$$

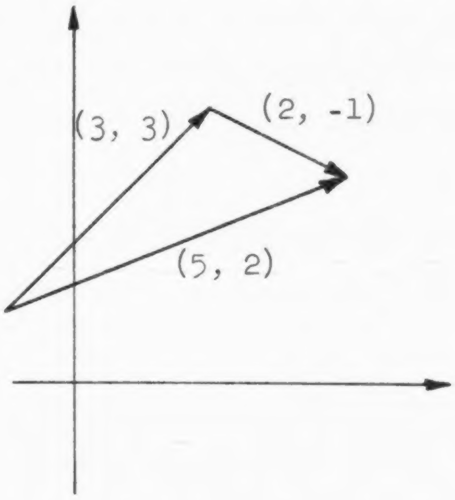
These conventions yield:

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} \quad \text{or} \quad (B - A) + (C - B) = C - A$$

$$B + \overrightarrow{BC} = C \quad \text{or} \quad B + (C - B) = C$$

If a coordinate frame is given, any displacement is a unique sum of an horizontal and a vertical displacement: $D = D_H + D_V$. Either of the uses of "+" described above translates to componentwise addition of number pairs. Turning this around: $(3,3) + (2,-1) = (5,2)$ interprets either as





Furthermore, $(5,2) - (3,3) = (2,-1)$ with either the point or displacement interpretation of "-" .

Now it is natural to say that constant velocity means displacement proportional to time, e.g., $F(t) - F(0) = t[F(1) - F(0)]$. Assuming a previous treatment of motion on a line (using displacements in the same way), and noting the relation above implies constant velocity motion of both the horizontal and vertical projections, we infer that this amounts to requiring these projections to be constant velocity motions.

In any event we get

$$F(t) = (a,b) + t(c,d)$$

which we proceed to analyze and interpret in problems. The velocity vector associated with this motion is of course the average displacement vector (c,d) .

It remains to postulate and make plausible the additivity of velocities, which guarantees that velocities behave like displacements. Any property with these characteristics is generally called a vector property (or, more usually, quantity).

Simple Harmonic Motion in General Topical Sequence

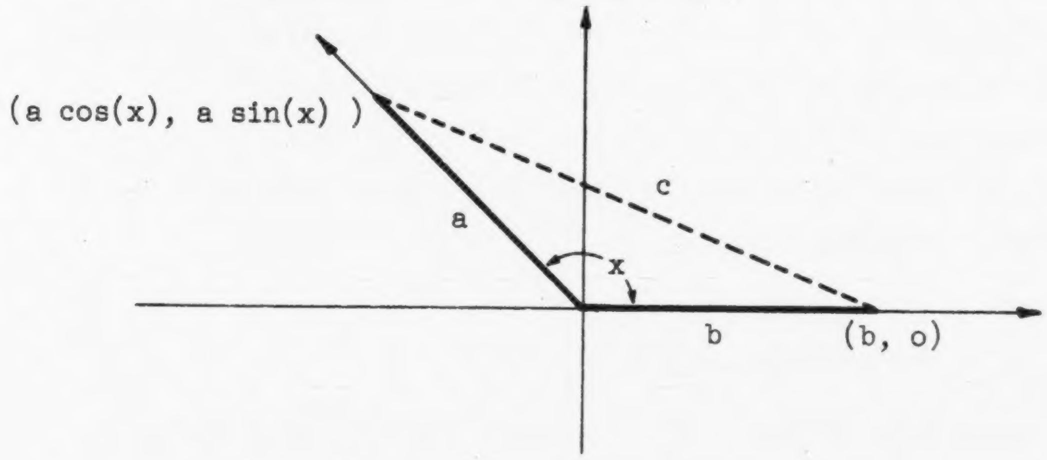
The suggested treatment is straightforward. Define cos and sin as coordinate functions associated with a standard c.c. (counter-clockwise) constant speed motion around central unit circle. Note relation with any earlier definition. Note arc length interpretation and make this basic in deriving properties. Derive position function for arbitrary uniform circular motion.

Read off elementary properties. Use constant speed and distance formula to get addition formula. Use addition formulas to compute some less obvious values.

Observe that unit length displacements are $(\cos(x), \sin(x))$ and develop vector formula for line:

$$L(t) = (a,b) + t(\cos(x), \sin(x)) .$$

Define the cosine of an (unoriented) angle with radian measure x as $\cos(x)$ and derive the law of cosines from



It remains to treat oriented angles and the law of sines.

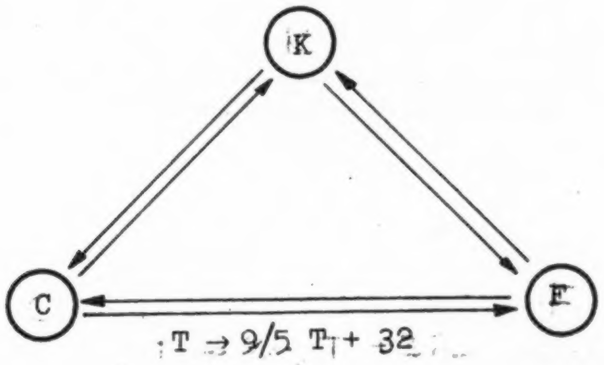
A Sampling of Problems for Sections 1-3 of General Topical Sequence

1. Linear Functions.

(1) What are some properties which can be coordinatized as a line? as a ray?

(2) If an object moving on a line has a position $F(t)$ in miles at time t in hours, what function G gives its position $G(t)$ in feet at time t in minutes?

(3) Fill in the temperature coordinate conversion functions:



(4) Find the function which converts atmospheric pressure from lbs. per sq. in. to atmospheres above 1 atmosphere.

(5) An object moving on a line has its position function converted from $t \to 6t - 5$ to $t \to 3t$ by a change of position scale. What is the change?

(6) In (5) change the position (time) scale so that the coordinate form of the position function is $t \to t$.

(7) If in a certain process pressure in some scale is proportional to absolute temperature, how is pressure related to temperature in centigrade scale?

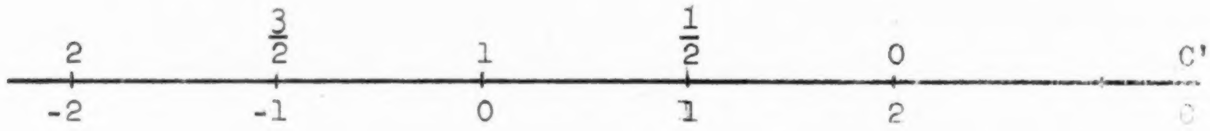
(8) Given a coordinate conversion function, $x \to ax + b$ how can you tell whether

(a) the positive direction is reversed?

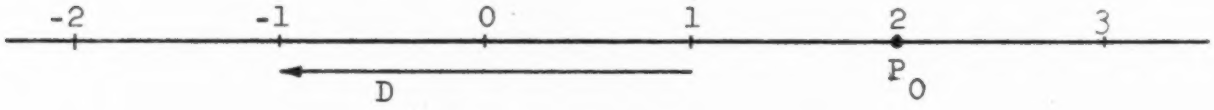
(b) the unit size is preserved?

(c) the origin is moved?

(9) Find formulas which convert C to C' and C' to C coordinates.



(10) If the unit-time displacement and initial position of an object moving with constant velocity are as shown, what is the position function?



2. Quadratic Functions.

(1) An object is thrown vertically up from the top of a 100 ft. tower with initial velocity 60 ft. per second. Find the coordinate form of its position function in case (a) time is given in seconds after the object is thrown and position is given in feet below the top of the tower; (b) same as (a) except position is given in feet above ground level; (c) time is given in seconds after the time of maximum height and displacement is in feet below the maximum height.

(2) In each of the cases in (1) what should be computed in order to find the time the object strikes the ground?

(3) A curve P is the graph of $x \to 3x^2 - 5x + 4$. Of what is P the graph if the origin is shifted 2 units to the right and 3 up?

(4) What displacement of the coordinate system converts $x \to 3x^2 + 6x + 5$ into $x \to 3x^2$?

(5) Is $x \to 3x^2 + 6x + 5$ of the form $x \to (ax + b)^2$?

(6) What quadratic functions are of the form $f(g(x)^2)$ where f and g are linear?

(7) The graph of a quadratic function has its vertex at (3,2) . What can be inferred about its formula? What can be inferred from the fact that the line of symmetry of such a graph contains (-1,0) ?

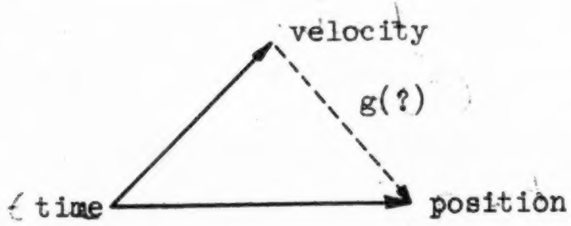
(8) In the situation of (1) find a formula for the time at which the object is at a given position during its descent.

(9) At what points does the graph of $x \to 3x^2 - 5x + 4$ intersect the horizontal line through

(a) (0,3) ,

(b) (0,1) ?

(10) In the situation of (1) show that for each velocity vector v there is a unique position $g(v)$ at which the object has velocity v by finding a formula for g .



3. Motion in a Plane.

(1) A ship moves with constant velocity (4,3) in certain time and position units and is at (2,-1) at time 1 . Find its position function. Find a coordinate description of its path.

(2) A ship moving with constant velocity is at (0,2) at time 1 and (5,0) at time 3 . Find its position function.

(3) Find the place where the paths of the ships in (1) and (2) cross. Do the ships collide? If not, find the minimum distance between them.

(4) If $F(t) = (a,b) + t(c,d)$ is a position function, under what conditions does the object move

(a) due east?

(b) due northeast?

(c) 30° south of east?

(d) with speed 1 ?

(5) An instrument on a moving plane measures the wind velocity as 500 mph from 30° to the left of the plane's heading. Predict

the wind gauge reading of the plane makes a 180° turn without altering engine power.

(6) Paint A is made by mixing 2 parts red with 3 parts blue. Paint B consists of 1 part red for every 4 parts blue. Give directions for using A and B to make paint C which has the formula

(a) 1 part red to 2 parts blue,

(b) equal proportions of red and blue.

What paints can be mixed from A and B ? Interpret geometrically.

(7) Standard projectile motion problems. Also find coordinate description of trajectory; e.g., if $F(t) = (10 + 20t\ ,\ -16t^2 + 100t + 20)$ think:

$$10 + 20t \xrightarrow{F_1^{-1}} t \xrightarrow{F_2} -16t^2 + 100t + 20$$

to get explicit description of trajectory.

(8) Find the position function for an object which moves uniformly clockwise around a circle of radius 3 at 2 revolutions every π time-units and which is at $(-1,0)$ at time 0 .

(9) From the earth the angle between Venus and Mars is 20° at a certain time. Venus is x and Mars y million miles from the earth. What is the distance between Mars and Venus?

Coordinates, Vectors, Functions: A Level 10 Course

The attached pages give a course outline of the first three chapters supplemented with some detail on the intended mathematical skeleton. This is a very slight revision of the early sections of the course sketched under the heading General Topical Sequence. This amplification is a response to expressed uncertainties as to the implementation of the original outline. A glimpse of the situations which can be discussed and the equipment to be developed to handle them may suggest a sequel.

Some of the themes which ought to play significant roles are:

1. Sources of functional correspondences.

2. The role of coordinate systems and measures in converting correspondences to number functions.
3. The change and lack of change in functions effected by coordinate change.
4. Geometric pictures and diagrams of functions and the way various properties of functions reflect in them.
5. The use of coordinates to reduce mappings into R^2 or R^3 to pairs or triples of number-valued mappings.
6. The various ways in which functions are used to describe (say) plane sets; as graphs, as ranges (parametrically), and as inverse images (level sets).
7. The general meaning of linear functions and their amenability to a complete, uniform analysis.
8. The new questions and difficulties which arise with non-linear functions.
9. The useful possibilities in extending number operations to functions; the ubiquity of the composition of functions.
10. The properties which a given function does or does not preserve. The search for functions with given properties, in particular Euclidean transformations.
11. The value of coordinates in translating geometric descriptions to number relations and in interpreting number relations geometrically.
12. Various ways in which problems may be translated into equivalent problems and the reasons therefor.

OUTLINE OF A LEVEL 10 COURSE

Chapter 1 - 3

Coordinates, Vectors, Functions

1. Linear Functions.

   1-1. Real numbers as coordinates for points on a line, points of time, states of temperature, and other properties. Linear functions as coordinate conversion functions.

   1-2. Effect of coordinate conversion on functions relating different properties as, e.g., a function giving the position of an object on a line in terms of time. Scale factor (a) and shift term (b) in linear function $x \to ax + b$.

   1-3. Line to line diagrams and graphs of linear functions. Inverses and composites.

   1-4. Displacements on a line, scalar multiplication, addition. Coordinates for displacements and addition of real numbers.

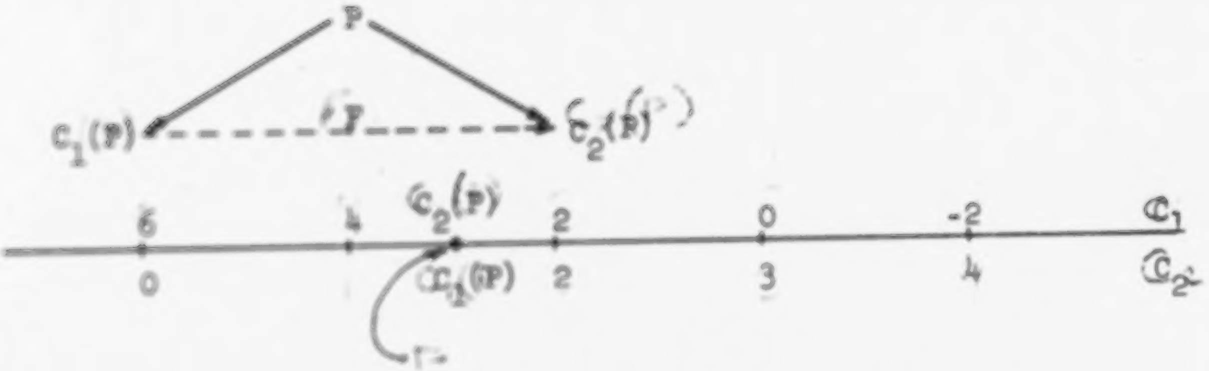
   1-5. Determination of isometries of the line, the opp function, similarities.

1A. Schema for Linear Functions.

Recall real numbers as line coordinates and review geometric significance of operations. Note that coordinate system involves arbitrary choices of origin and unit point. In fact, a coordinate system is a function

$$C : \text{points} \to \text{numbers}.$$

$C(P) = 2$ means that C assigns 2 to P . (There are reasons why a coordinate system might better be defined as the number to point map.) Since each such C is one-one, two coordinate systems C_1 , C_2 determine a number to number function F which converts C_1 to C_2 coordinates



Now F is linear and if $F(x) = ax + b$ then $|a|$ reflects the relation of units, $sgn(a)$ the relation of the orientations of C_1 and C_2, and b exhibits the relation of origins in terms of the C_2 system. Call a the scale factor and b the shift term.

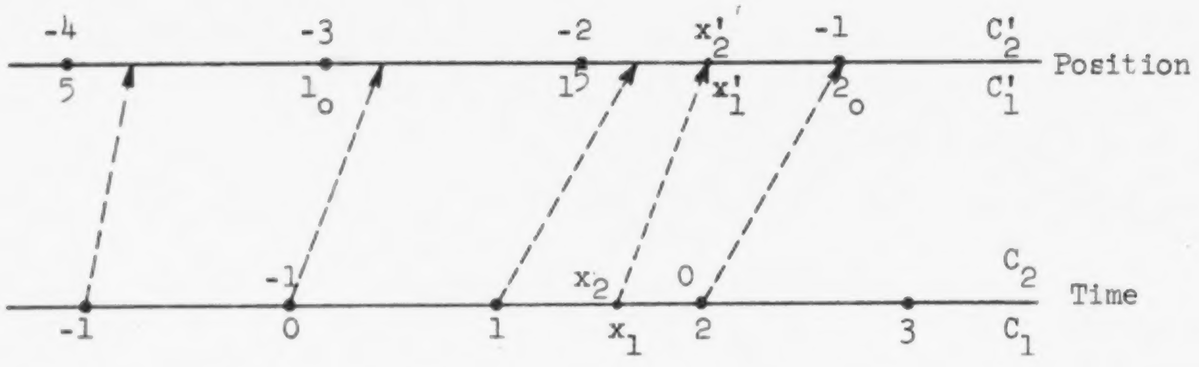
Linear functions arise also as relations between different properties which admit real number coordinates. The motion of an object on a line is specified by its position function, which assigns to each time the position of the object at that time. Each choice of time and line coordinates determines a coordinate form f of the position function.

$$\begin{tikzcd}
\text{time} \arrow[r] \arrow[d, shift right, "C"'] & \text{position} \arrow[d, "C'"] \\
t \arrow[u, shift right] \arrow[r, dashed, no head, "f"] & f(t)
\end{tikzcd}$$

Different choices of C or C' induce different coordinate forms f of the time-position function. If $f(t) = 30t$ in hours and miles then the coordinate form g corresponding to seconds and feet is given by

$$g(t) = 5280\, f(\frac{t}{3600}) .$$

Diagrams like this exhibit the four functions determined by a pair of coordinate systems for each property



If one coordinate form is linear then all are. For any given function relating one property to another it is always possible to choose coordinates so that the coordinate form is $x \to x'$. These considerations involve a discussion of composites and inverses. Each of these notions is best visualized in terms of line to line diagrams even if the function originates from consideration of a single line (i.e., property).

If the graph of a function f is $\{(x,y)|y = f(x)\}$ (often identified with f) , then every number to number function and every choice of plane coordinates determine a set of points which may be regarded as a picture of f . Linear functions have lines as pictures, and nonvertical lines have coordinate descriptions which are graphs of linear functions. The effect on the coordinate description of a given line of a plane coordinate change induced by horizontal or vertical coordinate changes can be computed.

The picture of f^{-1} can be described in terms of the picture of f .

Displacements on a line are mappings of the line to itself which in any coordinate system have the form $x \to x + a$ for suitable a . In a given system the number a serves to identify the displacement serving, if you like, as its coordinate. This idea extends readily to the plane where it is important to observe that composition of displacements corresponds to coordinatewise addition.

The isometries of a line can be described (in a series of exercises?) in terms of displacements and opp . Suitable coordinates give the standard forms $x \to -x$ and $x \to x + a$.

2. Quadratic Functions.

2-1. Falling objects, gravitational motion on a line. Sample time-position data, discussion of time-velocity function (as vector valued) and relation to position function, graphs and line to line diagrams.

2-2. Effects of coordinate changes on position and velocity functions, completing the square, simplification of descriptions of quadratic graph curves; e.g., $y = 3x^2 - 12x + 13$ to $y = 3x^2$, problems relating to objects projected vertically.

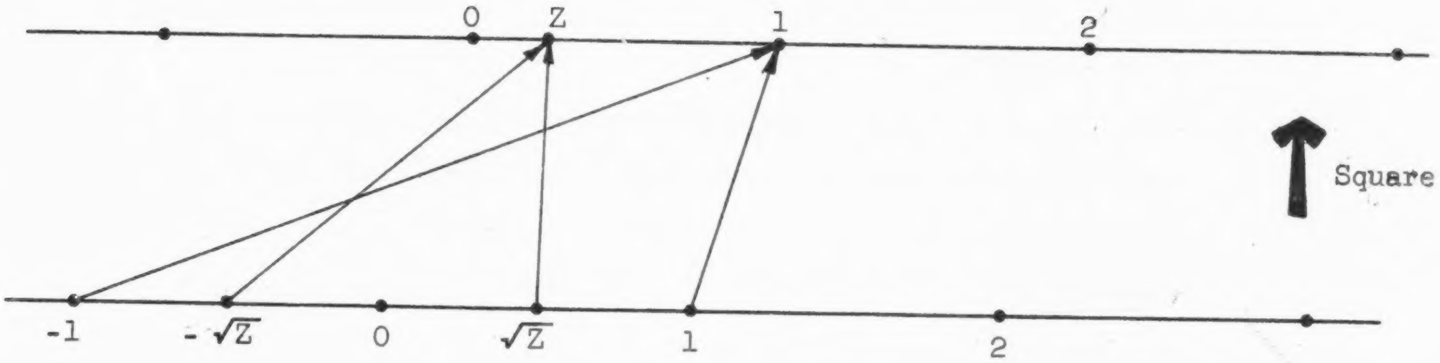
2-3. Finding the time at which an object moving vertically and gravitationally is at a given height. Resolution of the two-to-one property of quadratic functions by completing the square and resort to square roots. Solution of quadratic equations in terms of $x \to \sqrt{x}$. The computation problem, approximation of $\sqrt{x}$.

2A. Schema for Quadratic Functions.

One way to handle vertical gravitational motion is to offer sample time-position data and look for patterns, winding up with the conclusion that displacement from time t to $t + a$ is linear. Then $v(t)$ is defined as the average displacement in any time interval $t - a$ to $t + a$. The effect of initial conditions is analyzed by changing the time origin. This leads to the general relation between a position function and its associated velocity function.

Selecting coordinates in falling body problems and interpreting the resulting position function leads to the question of describing, say, the picture of $x \to 3x^2 - 12x + 13$ in simpler and more informative terms by selecting a plane coordinate system related to the curve. Selection of origin at the vertex yields the description $x \to 3x^2$. This in turn shows that the pictures of $x \to 3x^2$ and $x \to 3x^2 - 12x + 13$ in a single coordinate system are congruent. (We observe that plane displacements are isometries.)

The problem of inferring time from position calls attention to the two-to-one character of squaring. The line to line diagram of $x \to x^2$ suggests creation of the partial inverse $x \to \sqrt{x}$.



Previous work on completion of the square shows that this function is sufficient to describe inverse images under any quadratic function, i.e., to solve any quadratic equation.

Some discussion of the question: "does every positive number have a square root?" is in order, as is some treatment of the difference between

(1) $\sqrt{2}$ is the unique positive number x with $x^2 = 2$.

and

(2) $1.414 < \sqrt{2} < 1.415$.

3. Motion in a Plane.

3-1. Planar motion situations; ship sailing on a (pre-Columbian) sea, projectiles, moon, the notion of a time-position function whose values are points. Specification of point valued functions by coordinate functions: $f(t) = (f_1(t) , f_2(t))$.

3-2. Displacements and their coordinate description. Coordinate versions of addition and scalar multiplications of displacements. The standard correspondence between points and displacements.

3-3. Displacements derived from successive positions of a moving object. Constant velocity position functions and their coordinate form $t \rightarrow (a,b) + t(c,d)$, the associated velocity vector. The position function as a parametric description of the path (or track) of the object, the point and vector interpretations of $A + tB$. Conversion from parametric to $\{(x,y) | p(x,y)\}$ descriptions.

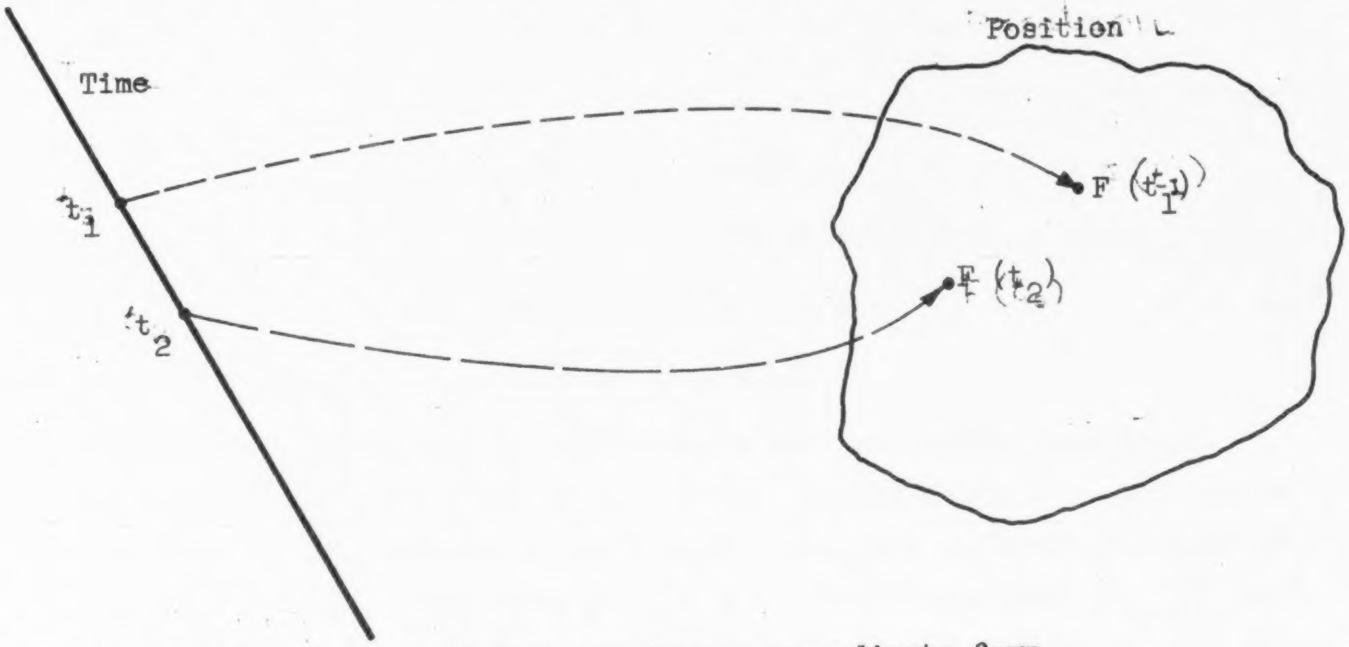
3-4. Planar gravitational motion, component motions and velocities, derivation of the position function $f(t) = (f_1(t) , f_2(t))$ where

f_1 gives constant velocity motion and f_2 gravitational motion on a line. Derivation of usual description of path and posing of questions about time, position and velocity.

3-5. Uniform circular motion. Analysis of examples of position functions. Standard forms of coordinate functions for uniform circular motion: $W(t) = (\cos(t) , \sin(t))$. Elementary properties of cos and sin , addition formulas, all derived from winding function W . What is the velocity function for u.c.m. like (as a plane point to plane vector mapping)?

3A. Schema for Motion in a Plane.

To specify motion in a plane is to associate with each point in time the point in the plane representing the position of the object at that time. Thus the time-position function has a line to plane diagram.



Coordinatizing line and plane produces a coordinate form $f(t) = (f_1(t) , f_2(t))$ thus reducing the description to a pair of number to number functions.

Each pair of positions determines a displacement which, as on the line, can be identified by coordinate differences. These displacement coordinates are unaffected by a change of origin. From the decomposition of a displacement into horizontal and vertical components the coordinate form of displacement composition (i.e., "addition") follows. (For any real t , tD can be defined by a betweenness stipulation.)

The map $O \to D(O)$ permits the identification of vectors with points, hence of sets of vectors with sets of points. The natural meaning of constant velocity is "position displacements proportional to time displacements". Thus $\overrightarrow{F(t_1)F(t_2)} = (t_2 - t_1)V$ which leads to the coordinate form

$$f(t) = P + tV = (x_1,x_2) + t(v_1,v_2)$$

and the identification of V as the velocity vector associated with f. For later purposes it should be observed that motions of the coordinate axis projections of a constant velocity motion have velocities which sum to the constant velocity.

The set of all positions (call it path) of an object with position function f is $\{P|P = f(t) \text{ for some } t\} = S$. A description of this type is called a parametric description of S. It amounts to specifying a function whose range is S. A position function describes the path parametrically. For constant velocity motions this can be converted to the usual description by observing that

$$f_2(f_1^{-1}(f_1(t))) = f_2(t) .$$

Thus if $f(t) = (2t - 3 , 6t - 5)$ then

$$6t - 5 = 6[\tfrac{1}{2}(2t - 3) + \tfrac{3}{2}] - 5$$

or

$$f_2(t) = 3f_1(t) + 4 .$$

So

$$S = \{(2t - 3 , 6t - 5)\}$$

$$= \{(x,y)|y = 3x + 4\} .$$

Experiments which show the independence of horizontal and vertical projections of a gravitational motion can be cited and their significance for the position function examined. From these considerations the conclusion that f_1 is constant velocity and f_2 gravitational follows. The fact that f_1 is linear permits the straightforward derivation of the usual description of the path.

It should be noted that the addition formulas for cos and sin are consequences of the assumption that equal time intervals are mapped onto congruent arcs.

Sample of An Unorthodox Analysis Semester

1. Decimal Expansions.

   1-1. Infinite decimals as point specifications; the nested interval principle, decimals as limits.

   1-2. Partial sum sequence of a sequence, the monotonicity principle; binary numerals; computation with truncated decimals.

   1-3. Geometric series, partial sums and remainder, polynomial approximation to $\frac{1}{1-x}$.

2. Exponential Processes.

   2-1. Geometric series and area under $x \to 2^{-x}$ from 0 to n and 0 to α .

   2-2. Interest, compounding, the limiting case $(1 + \frac{r}{n})^n \to ?$

   2-3. Rate of change in exponential processes; $\exp_a'(x) = \exp_a'(0) \cdot \exp_a(x)$.

   2-4. Relation of rates to linear approximation, tangents.

   2-5. Log' via reflection of exp ; area under $\frac{1}{x}$ in terms of rectangular approximation; pinching principle, error term; $\log x \to \alpha$ as $x \to \alpha$, harmonic series.

3. Local Approximation.

   3-1. Linear interpolation approximation (log, exp, sin, $\sqrt{\ }$) ; convexity and its implications for error.

   3-2. Local linear approximation using derivative; attempt to compute e^x using successive approx.;

   $(e^x > x + 1 \to \int_1^x e^t dt > \int_1^x (t+1)\, dt)$;

   approx. to $\frac{1}{1-x}$ via partial fractions;

   approx. to $\log(x)$ via integral;

problem of points of convergence;
derivatives of polynomials, local approx. as derivative matching.

3-3. Global Approximation; polynomial interpolation, Simpson's rule; interpolation formula, curve fitting.

- - - - - - - - -

An alternative to the above is to deal with motion in the plane via vector-valued functions and their derivatives, analyzing gravitational and circular motion and studying first and second order linear differential equations.

Systems of Sentences in Two or More Variables for Grades 10-12

It is recommended that the following material not be included in the 7-9 sequence.

Background:

1. In Grade 8 students will have studied systems of first degree sentences in two variables with a slight introduction to linear programming. (Chapter 12, 1967 sequence.)

2. In Grade 8 students will have studied the quadratic function $f : x \rightarrow a(x - h)^2 + k$ extensively and be familiar with the equation of a circle. (Grade 7, Chapter 2, 1967 sequence.)

3. They will not have any experience with equations of the hyperbola or ellipse, and will not be familiar with the general second degree sentence in two variables.

4. In Grade 9, Chapter 3, students will develop, hopefully, a higher level of sophistication in working with first degree sentences in two variables as they develop the linear programming chapter.

I recommend that a study of systems of sentences like:

$$\begin{cases} x + 2y - 3 = 0 \\ x^2 - 3x + 5 = y \end{cases}, \qquad \begin{cases} 2x^2 - 3x + 4 + y \leq 0 \\ x + 2y - 5 = 0 \end{cases},$$

$$\begin{cases} x^2 + y^2 - 3 = 0 \\ 2x^2 - 3x + 4 - 5y^2 = 0 \end{cases}, \qquad \vdots$$

along with their graphical representation be studied in conjunction with the appropriate sections in the 10-12 sequence.

I also recommend that systems of sentences like

$$\begin{cases} x + y + z - 3 = 0 \\ 2x - 5y + 7z + 1 = 0 \\ 5x - 2y - 3z + 10 = 0 \, , \end{cases} \qquad \begin{cases} x + y + z + w - 4 = 0 \\ 2x - 5y + z - 3w + 10 = 0 \\ 5x + 10y - 7z + w - 2 = 0 \quad \cdots \\ x + y - z - w + 2 = 0 \, , \end{cases}$$

be included at appropriate places in the 10-12 sequence.

Some reasons for the recommendations:

1. Students will have matrices to handle systems of first degree equations in two or more variables in the 10-12 sequence.
2. It seems more appropriate to study systems of second degree sentences in two variables when a knowledge of all of the conic sections, and some knowledge of transformations in the plane are available to the student.
3. Study of systems of equations in the 10-12 sequence can arise naturally in the spiral of the "stream" of modeling and linear programming.

Outline of Elementary Functions and Calculus Course

0. Introduction.

1. Sequences, Series, Mathematical Induction.

   Examples of sequences.

   Inventing general term.

   Definition (functions on positive integers).

   Graphs of sequences.

   Questions about behavior.

      Bounded or unbounded.

      Maximum and minimum values.

      Increasing, decreasing, constant, oscillating.

      Limiting value.

   Defition of limit.

      Proof that $\frac{1}{n} \to 0$.

   Use of theorems (without proof) about limits of sum, product, etc., to evaluate limits of sequence.

   Informal induction to find

   $$1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

   $$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)}{2} \cdot \frac{2n+1}{3}$$

   $$1^3 + 2^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$$

   Infinite series.

   Telescoping series.

   Mathematical induction.

   Arithmetic and geometric series.

   Convergence and divergence for geometric series.

2. Functions on Reals.

   Definitions of functions with examples.

   Graphs of functions.

   Algebraic combinations of functions.

Linear and quadratic functions.
Polynomial and rational functions.

3. Polynomial Functions.

Review of synthetic substitution.
Graphs, relative extrema, roots.
Slope of tangent and derivatives.
Continuity and differentiability.
Derivatives of sums, products and powers (by induction from products).
Rolle's Theorem: $Df(x) = 0 \longrightarrow f(x) = c$.
Mean Value Theorem.

4. Applications of Derivatives.

Tangent and normal lines.
Extreme values.
Maximum and Minimum problems.
Concavity Points of Inflexion.
Higher derivatives.
Newton's method for roots.
Rates of change, velocity, acceleration.
Differentials and approximation.

5. Area and Volume.

Area properties.
Area under parabola.
    Upper and lower sums.
Limits of upper and lower sums.
Definite integral.
Additivity of areas.
Antiderivatives and Fundamental Theorem.
Area between curves.
Volume problems.
Motion problems.
Work problems.
Average values.

6. Theory of Limits.

7. Rational Functions.

Curve tracing.
Derivatives (quotient rule).
Negative integral powers.
Extrema.
Vertical and horizontal asymptotes.
Simple areas (show how easily we get stuck).

8. Composite Functions.

Composite functions (using three parallel lines).
Chain rule using derivatives as local multipliers.
} See the Derivative as Local Multiplier, pp. 197.

Inverse functions.
Derivatives of inverse functions.
Root functions and their derivatives.
    Review of fractional exponents.
Differentiation of implicit functions.

9. Exponential and Log Functions.

ln x as area under $y = \frac{1}{x}$.
Graph of $\ell n\ x$.
Proof that $\ell n\ ax = \ell n\ a + \ell n\ x$ and other logarithmic properties.
Polynomial approximation.

$$\frac{1}{1+u} = 1 - u + u^2 - u^3 + \frac{u^4}{1+u}$$

$$\ell n\ (1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + E$$

$$\left(0 < E < \frac{u^5}{5} \right) .$$

Computation of a few logs.
Exponential as inverse of log.
Property $e^{a+b} = e^a \cdot e^b$.
Derivatives of e^x and e^{kx} .
Polynomial approximation to e^x .
Computation of values of e^x .

Growth and Decay.

Applications of $Dy = ky$.

Other bases.

10. Circular Functions.

Definition (functions of arc length).

Periodicity.

Graphs of $\sin x$, $\cos x$.

Angle and angle measure.

Vectors and rotation.

Addition formulas.

Identities.

Derivatives of $\sin x$, $\cos x$, $\sin kx$, $\cos kx$, $\tan kx$.

Antiderivatives.

Inverse circular functions.

Simple harmonic motion $(D^2y + k^2y = 0)$.

Polynomial approximations to $\sin x$, $\cos x$, $\arctan x$.

Computation of π .

Two Questions:

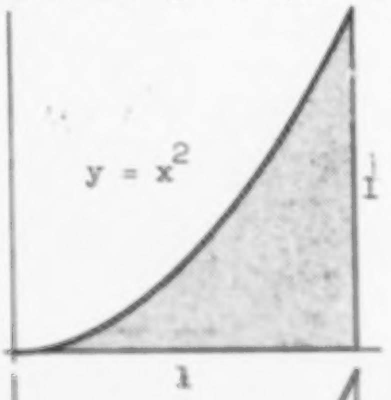
1. Is it desirable at some stage to prove theorems for limit of sum, product, quotient? If so, how rigorously?
2. Is it desirable to have a final chapter or set of exercises which stress overall comprehension?

A Sketch of an Introductory Chapter to Elementary Functions and Calculus

Chapter 0: Introduction

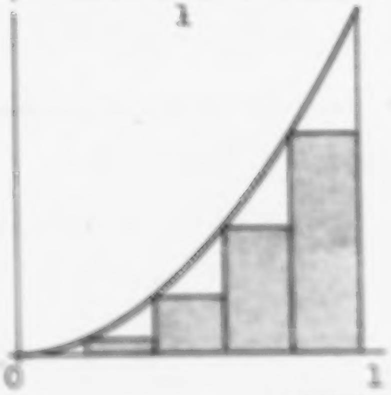
What is calculus about? We can give some idea by discussing two typical problems.

Problem 1. To find the area under the curve $y = x^2$ and above the interval $[0,1]$.

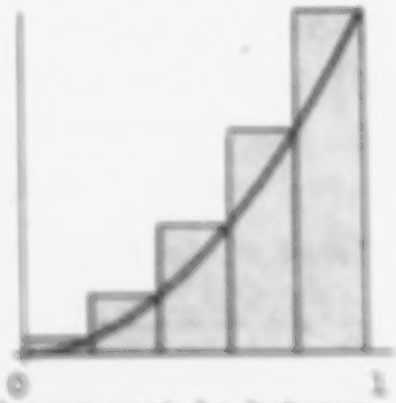


In geometry, we learn how to find the area of a rectangle, of a triangle and, more generally, the area of any polygon. Here we meet a new kind of problem because the upper boundary is curved.

It is natural to approximate the figure by putting together rectangles as shown. By adding the areas of these rectangle we obtain a result (called a lower sum) which is too small.



In a similar way we may add the areas of rectangles to obtain an upper sum, that is, a sum which is too large.



For a given subdivision of the base, the required area must be between the corresponding upper and lower sums.

If we use a very fine subdivision of the base -- into 1000 equal parts, say -- it seems clear that the upper and lower sums will be very close together. We can now ask whether as we choose finer and finer subdivisions the upper and lower sums approach a common value. If so, this value will represent the required area exactly.

To investigate this possibility, we need to know how to add a tremendous list of numbers. To begin with let us divide $[0,1]$ into tenths. The upper sum is

$$.001 + .004 + .009 + \dots + .100 \ ;$$

that is,

$$.001\,(1 + 4 + 9 + \dots + 100)$$

or

$$\frac{1}{10^3}\,(1 + 2^2 + 3^2 + \dots + 10^2)\ .$$

If the interval is divided into hundredths, the sum is

$$\frac{1}{100^3}\,(1^2 + 2^2 + \dots + 100^2)\ .$$

If we go on to thousandths, ten thousandths and so on, the arithmetic becomes truly frightening.

What we need is a general result for any number n of rectangles, that is, for the upper sum

$$S_n = \frac{1}{n^3} + \frac{2^2}{n^3} + \frac{3^2}{n^3} + \dots + \frac{n^2}{n^3}$$

or

$$S_n = \frac{1}{n^3}\,[1^2 + 2^2 + 3^2 + \dots + n^2]\ .$$

If we had such a result, we could substitute $n = 10$, 100 , 1000 and so on. We might even be able to see what the limiting value of the upper sum is as we increase n beyond all bounds.

Later we shall obtain such a result and shall find that

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}\ ,$$

so that

$$S_n = \frac{n(n+1)(2n+1)}{6n^3} = \frac{1}{6}\,(1+\frac{1}{n})(2+\frac{1}{n})\ .$$

What happens to S_n as n becomes very large? $1 + \frac{1}{n}$ will be extremely close to 1 and $2 + \frac{1}{n}$ very close to 2 . Then S_n will be very nearly equal to $\frac{1}{3}$. Actually S_n will be as close to $\frac{1}{3}$ as we please for sufficiently large values of n. Since a similar result is obtained by studying lower sums we conclude that the required area is $\frac{1}{3}$.

The limit process described is typical of the calculus and may be used to find not only the areas of figures with curved boundaries, but the volumes of space figures with curved boundaries, the energy required to lift a rocket to a height of 4000 miles and the solution to many similar problems. The idea of limit is the central one. A study of this concept leads us to astonishingly simple solutions of many important problems. It will be found for example that the problem just discussed can be solved in a few lines, after we have reached a certain stage in our understanding.

Problem 2. Another problem which can be solved by the use of limits is that of finding the speed of a body dropped from rest. In physics we learn that the distance d feet fallen in t seconds is given by the formula

(1) $\quad d = 16t^2$.

(Actually this is an approximation. A better approximation is given by the formula $d = 16.11t^2$.)

As everyone knows, the body goes faster as it continues to fall. Suppose that we wish to know exactly how fast it is going when $t = 2$.

It is easy enough to find the average speed over an interval beginning at $t = 2$, for example the interval between $t = 2$ and $t = 3$.

In fact at $t = 2$, $d = 16 \times 4 = 64$,

and at $t = 3$, $d = 16 \times 9 = 144$.

The body has travelled $144 - 64 = 80$ ft. in one second. Therefore its average speed during this second is 80 ft./sec.

If we consider a shorter time interval, say that between $t = 2$ and $t = 2.1$ we obtain a result which is closer to what we desire. As before,

when $t = 2$, $d = 64$

when $t = 2.1$, $d = 16 \times (2.1)^2 = 70.56$.

In $\frac{1}{10}$ of a second, the body has fallen 6.56 feet. Over this interval, the average speed is $\frac{6.56}{.1} = 65.6$ ft./sec.

What would happen if we averaged over .01 of a second? .001 of a second?

As in the case of area, we need a general result to rescue us from the arithmetic.

Let us consider the time interval from $t = 2$ to $t = 2 + \frac{1}{n}$. The corresponding distances are

$$64$$

and

$$16\,(2+\frac{1}{n})^2 = 16\,(4 + \frac{4}{n} + \frac{1}{n^2})$$

$$= 64 + \frac{64}{n} + \frac{16}{n^2} .$$

The distance covered in the time interval $\frac{1}{n}$ is $\frac{64}{n} + \frac{16}{n^2}$.

Dividing $\dfrac{\frac{64}{n} + \frac{16}{n^2}}{\frac{1}{n}} = 64 + \frac{16}{n}$

gives the average speed between $t = 2$ and $t = 2 + \frac{1}{n}$.

This average speed is

$$(64 + \frac{1}{n}) \text{ ft./sec.}$$

The limiting value is 64 ft./sec.

This is what we take to be the actual speed at $t = 2$.

Again we have used a limit process to obtain the required result.

Chapter 6: Theory of Limits

The following indicates how the theory of limits might be handled in the course outlined for Elementary Functions and Calculus.

The theorems about sum, product and quotient sequences would be stated and used in Chapter 1, where only a very informal justification will be given.

These theorems would be used again to find the slopes of tangents in Chapter 3, and also in Chapter 5 on Areas.

At some point, these theorems should be proved. We suggest that Chapter 6 is an appropriate place. We have included here:

1. A discussion of the limit of a sequence suitable for Chapter 1.
2. Slopes of Tangents via Sequences. (See also Chapter 0.)
3. Proofs of Limit Theorems for Sequences suitable for Chapter 6. Section 3-1 on Null Sequences has been written. Section 3-2 has been left in outline form.
4. The Derivative as Local Multiplier. (Referred to in the Outline under Chapter 8.)

1. The Limit of a Sequence.

Consider the sequence $1, \frac{3}{2}, \frac{4}{3}, \ldots$ whose general term is $1 + \frac{1}{n}$. We say that $1 + \frac{1}{n}$ approaches 1 as a limit and write

$$1 + \frac{1}{n} \to 1 .$$

This means that the difference between $1 + \frac{1}{n}$ and 1 (which is $\frac{1}{n}$) is ultimately as small as we please. By "ultimately" we mean "for all values of n which are large enough".

For example if we wish to have $\frac{1}{n} < \frac{1}{1000}$, it is sufficient to choose $n > 1000$. It is useful to have a letter to represent a number as small as we please. The Greek letter ϵ is the customary one. In the example, $\epsilon = \frac{1}{1000}$. It is also helpful to have a letter to tell us how large a value of n we must choose to guarantee the degree of closeness ϵ. We use N for this purpose. In the example, $N = 1000$ since

$$\frac{1}{n} < \frac{1}{1000} \ (=\epsilon) \quad \text{if} \quad n > 1000 \ (=N) .$$

In general we write

$$\frac{1}{n} < \epsilon \quad \text{if} \quad n > N .$$

If we desire that $\frac{1}{n}$ shall be less than $\frac{1}{1,000,000}$, we are not surprised to find that we must go further out. In fact

$$\frac{1}{n} < \frac{1}{1,000,000} \quad \text{if} \quad n > 1,000,000$$

so that in this case N is $1,000,000$.

It is easy to see that more generally, for any positive ϵ

$$\frac{1}{n} < \epsilon \quad \text{if} \quad n > \frac{1}{\epsilon}$$

so that $N = \frac{1}{\epsilon}$ tells us in each case how far out we must go to be sure that all terms are indeed "as small as we please".

Suppose that we have any sequence

$$x_1 , x_2 , x_3 , \dots , x_n , \dots$$

which has a limit, say a . In the language that we have just learned this fact may be expressed as follows:

For any positive number ϵ , however small, there is a corresponding whole number N so that

(1) $$|x_n - a| < \epsilon \quad \text{if} \quad n > N .$$

$|x_n - a|$ is the distance from x_n to the limit a . We use the absolute value sign (which was unnecessary in our example) to take care of cases in which some or all values x_n are less than a .

For example, in the sequence

$$0 , \frac{3}{4} , \frac{8}{9} , \frac{15}{16} , \dots , 1 - \frac{1}{n^2} , \dots$$

all terms are less than the limit 1 . The distance of $x_n = 1 - \frac{1}{n^2}$ from 1 is $1 - (1 - \frac{1}{n^2})$ not $(1 - \frac{1}{n^2}) - 1$. But

$$|x_n - 1| = |(1 - \frac{1}{n^2}) - 1| = 1 - (1 - \frac{1}{n^2}) = \frac{1}{n^2} .$$

So in this example, (1) becomes

(2) $$\frac{1}{n^2} < \epsilon \quad \text{if} \quad n > N .$$

We must show that for any given positive number ϵ , we can find N so that $\frac{1}{n^2} < \epsilon$ is guaranteed to be correct for $n > N$. It is easy to do this.

If $$\frac{1}{n^2} < \epsilon$$

it must be true that

$$1 < n^2 \epsilon$$

and therefore

$$\frac{1}{\epsilon} < n^2 .$$

We can write this more conveniently as

$$n^2 > \frac{1}{\epsilon} .$$

Finally it must be the case that

$$n > \frac{1}{\sqrt{\epsilon}} .$$

Going backwards, if n is indeed greater than $\frac{1}{\sqrt{\epsilon}}$, $\frac{1}{n^2}$ is surely less than ϵ as required. The expression $\frac{1}{\sqrt{\epsilon}}$ tells us what N is. If $\frac{1}{\sqrt{\epsilon}}$ should be a whole number, $N = \frac{1}{\sqrt{\epsilon}}$; otherwise we can take N to be the whole number part of $\frac{1}{\sqrt{\epsilon}}$.

For example, if we require that the distance between x_n and 1 be less than $.0001$ (that is, if we set $\epsilon = \frac{1}{10,000}$)

N is $\frac{1}{\sqrt{\epsilon}} = \frac{1}{\sqrt{\frac{1}{10,000}}} = \frac{1}{\frac{1}{100}} = 100$.

That is, for all $n > 100$, x_n will be within $.0001$ of the limit 1 ,

if $\epsilon = \frac{1}{100,000}$

$\frac{1}{\sqrt{\epsilon}} = \frac{1}{\frac{1}{100\sqrt{10}}} = 100\sqrt{10} \approx 316.2$. Hence $N = 316$.

This means that in our sequence x_n is closer to 1 than $.00001$ for all terms beyond the 316th one.

2. Slopes of Tangents via Sequences.

To study $y = x^2$ near $x = 2$, $y = 4$

let $$x_n = 2 + \frac{1}{n}$$

$$y_n = 4 + \frac{4}{n} + \frac{1}{n^2} .$$

Then $$x_n \to 2$$

and $$y_n \to 4 .$$

$$m_n = \frac{y_n - 4}{x_n - 2} = \frac{\frac{4}{n} + \frac{1}{n^2}}{\frac{1}{n}} = 4 + \frac{1}{n}$$

is the slope of the secant line joining $(2,4)$ to (x_n, y_n) .

This slope $m_n \to 4$ as n becomes infinite.

Similarly with $x_n = 2 - \frac{1}{n}$.

Generalizing,

let $x_n = 2 + z_n$ where z_n is an arbitrary null sequence with no 0 member.

Then $$y_n = 4 + 4z_n + z_n^2 .$$

As $$x_n \to 2 , \ y_n \to 4 .$$

This expresses "continuity" at $x = 2$.

$$m_n = \frac{y_n - 4}{x_n - 2} = \frac{4z_n + z_n^2}{z_n} = 4 + z_n \to 4 ,$$

the slope of the tangent at $x = 2$.

It is easy to generalize to an arbitrary point (a, a^2) on the curve and to treat other curves in an analogous fashion.

We shall say that $f(x) \to b$ as $x \to a$ if for an arbitrary sequence $\{x_n\}$ of x_n's which are different from a and lie within the domain of f , the corresponding sequence $\{y_n\} = \{f(x_n)\}$ converges to b . The theorems to be proved for limits of sequences then become theorems about limits of functions on the reals.

3. Proofs of Limit Theorems for Sequences.

3-1. Null Sequence Theorems.

A particularly simple kind of sequence is one whose limit is 0. Such a sequence is called a null sequence because "null" is a name for zero.

$$1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$$

$$1, \frac{1}{4}, \frac{1}{9}, \ldots, \frac{1}{n^2}, \ldots$$

$$-1, -\frac{1}{4}, -\frac{1}{9}, \ldots, -\frac{1}{n^2}, \ldots$$

$$1, -\frac{1}{2}, \frac{1}{3}, \ldots, \pm\frac{1}{n}, \ldots$$

$$1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \ldots, \frac{1}{\sqrt{n}}, \ldots$$

are all examples of null sequences.

We express this fact by writing $\frac{1}{n} \to 0$, $\frac{1}{n^2} \to 0$, $\frac{-1}{n^2} \to 0$, $\pm\frac{1}{n} \to 0$ and $\frac{1}{\sqrt{n}} \to 0$.

We shall use z_n for the nth term of a null sequence because the letter z suggests zero.

If we are discussing two different null sequences we shall use z_n and w_n to represent their nth terms.

We know that $\frac{1}{n} \to 0$ and $\frac{1}{n^2} \to 0$. Can we conclude that $\frac{1}{n} + \frac{1}{n^2} \to 0$? The sequence whose general term is $\frac{1}{n} + \frac{1}{n^2}$ begins

$$1 + 1, \frac{1}{2} + \frac{1}{4}, \frac{1}{3} + \frac{1}{9}, \ldots$$

that is

$$2, \frac{3}{4}, \frac{4}{9}, \ldots$$

Suppose that we choose $\epsilon = \frac{1}{1000}$ and wish to show that

(1) $$\frac{1}{n} + \frac{1}{n^2} < \frac{1}{1000} \quad \text{for } n \text{ large enough.}$$

This will surely be the case if

(2) $$\frac{1}{n} < \frac{1}{2000}$$

and

(3) $$\frac{1}{n^2} < \frac{1}{2000} .$$

(2) is true if $n > 2000$ and

(3) is true if $n^2 > 2000$, that is, if

$$n > \sqrt{2000} = 44.72 .$$

If we choose $N = 2000$, the larger of the numbers 2000 and 44, (2) and (3) are both true and hence (1) is satisfied.

This example illustrates an important theorem.

Theorem 1. If $s_n \to 0$ and $t_n \to 0$

then $s_n + t_n \to 0$.

Proof: We must show that given any $\epsilon > 0$, there is a positive integer N so that

$$|s_n + t_n| < \epsilon \quad \text{if} \quad n > N .$$

Since

(4) $$|s_n + t_n| \leq |s_n| + |t_n|$$

it is sufficient to show that $|s_n|$ and $|t_n|$ are each less than $\frac{\epsilon}{2}$ when n is sufficiently large.

Now since $s_n \to 0$

$$|s_n| < \epsilon' \quad \text{if} \quad n > N' ;$$

and since $t_n \to 0$,

$$|t_n| < \epsilon' \quad \text{if} \quad n > N'' .$$

If we take $\epsilon' = \frac{\epsilon}{2}$ and $\epsilon'' = \frac{\epsilon}{2}$ and if N is the larger of the two numbers N' and N'' we have from (4)

$$|z_n + w_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{if} \quad n > N .$$

This completes the proof.

Consider the following sequences

$$1 , \frac{1}{2} , \frac{1}{3} , \dots , \frac{1}{n} , \dots$$

and

$$-1 , -\frac{3}{2} , -\frac{5}{3} , -\frac{7}{4} , \dots , -2 + \frac{1}{n} , \dots$$

If $z_n = \frac{1}{n}$ and $x_n = -2 + \frac{1}{n}$ what can we say about the sequence whose general term is $x_n z_n$? It is not hard to guess that $x_n z_n \to 0$. Let us prove that this is true.

We see that $\qquad -2 < x_n < 0$

so that $\qquad |x_n| < 2$

Now $\qquad |x_n z_n| = |x_n| \; |z_n| < 2|z_n| = \frac{2}{n}$.

Since $\qquad |x_n z_n| < \frac{2}{n}$

we can make $|x_n z_n| < \epsilon$ by choosing $\frac{2}{n} < \epsilon$ which means that $n > \frac{2}{\epsilon}$.

This is an example of a second important theorem.

Theorem 2. If $z_n \to 0$ and $|x_n| \leq C$ $(C > 0)$ then $x_n z_n \to 0$.

In the example, $z_n = \frac{1}{n}$, $x_n = -2 + \frac{1}{n}$ and $C = 2$.

Proof: We must show that given any $\epsilon > 0$, an N may be chosen so that

$$|x_n z_n| < \epsilon \quad \text{if} \quad n > N .$$

We know that

$$|x_n z_n| = |x_n| \; |z_n| \leq C \; |z_n| .$$

Since $z_n \to 0$ we know that for every $\epsilon' > 0$, there is a corresponding N for which

$$|z_n| < \epsilon' \quad \text{if} \quad n > N .$$

Then $|x_n z_n| < C\epsilon'$ if $n > N$.

If we choose $\epsilon' = \frac{\epsilon}{C}$

then

$$|x_n z_n| < \epsilon \text{ if } n > N .$$

This completes the proof.

Example: $\frac{1}{n^2+n^3} = \frac{1}{n^2} \cdot \frac{1}{1+n}$. Since $\frac{1}{n^2} \to 0$ and $\frac{1}{1+n} < 1$, the theorem applies.

Theorem 2 has two corollaries.

Cor. A. If $z_n \to 0$ and $w_n \to 0$, then $z_n w_n \to 0$.

Proof: Since w_n approaches a limit it is bounded (see Chapter 1). That is, $|w_n| < C$ for some positive C . The theorem then applies.

Examples:

1. Since $\frac{1}{n} \to 0$ and $\frac{1}{n} \to 0$, $\frac{1}{n} \cdot \frac{1}{n} \to 0$,

   that is, $\frac{1}{n^2} \to 0$.

2. Since $\frac{1}{n} \to 0$ and $\frac{1}{n^2} \to 0$, $\frac{1}{n} \cdot \frac{1}{n^2} \to 0$,

   that is, $\frac{1}{n^3} \to 0$.

Cor. B. If $z_n \to 0$, $kz_n \to 0$ where k is any constant, positive, negative or zero.

Proof: Consider the constant sequence

$$k , k , k , \dots , k , \dots$$

with $x_n = k$. Theorem 2 applies with $C = |k|$, so long as $k \neq 0$.

However if $k = 0$, $kz_n = 0$ and we have the sequence

$$0 , 0 , 0 , \dots , 0 , \dots$$

which is surely a null sequence.

Example:

Since $\frac{1}{n^2} \to 0$, $\frac{4}{n^2} \to 0$

$-\frac{3}{n^2} \to 0$, etc.

3-2. More General Limit Theorems.

Definition. $x_n \to a$ means $x_n - a \to 0$.

(1) $x_n \to a$, $y_n \to b \Longrightarrow x_n + y_n \to a + b$.

Proof: $x_n - a \to 0$, $y_n - b \to 0$ [Def.]

$x_n - a + y_n - b \to 0$ [1]

$(x_n + y_n) - (a+b) \to 0$

$x_n + y_n \to a + b$ [Def.]

(2) $x_n \to a \Longrightarrow kx_n \to ka$

Proof: $x_n - a \to 0$

$k(x_n - a) \to 0$ [Cor.(B)]

$kx_n - ka \to 0$

$kx_n \to ka$ [Def.]

(3) $x_n \to a$, $y_n \to b \Longrightarrow x_n y_n \to ab$

Proof: $x_n - a \to 0$ [Def.]

$y_n - b \to 0$ [Def.]

$x_n y_n - ay_n - bx_n + ab \to 0$ [Cor.(A)]

$x_n y_n - ay_n - bx_n \to -ab$. [Def.]

But $bx_n \to ab$, [2]

and $ay_n \to ab$. [2]

Hence $x_n y_n \to ab$. [1]

(4) $x_n \to 1 \Longrightarrow \frac{1}{x_n} \to 1$

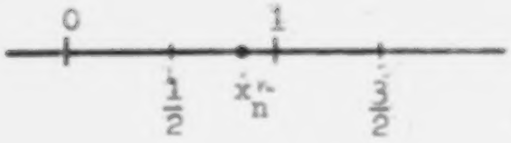
Proof: $x_n - 1 \to 0$ [Def.]

For n sufficiently large, say $n > N_1$,

$$|x_n - 1| < \frac{1}{2} .$$

Then $-\frac{1}{2} < x_n - 1 < \frac{1}{2}$

$$\frac{1}{2} < x_n < \frac{3}{2} .$$



In particular,

$$x_n > \frac{1}{2}$$

and $\frac{1}{x_n} < 2 .$

Now since $x_n - 1 \to 0$

and $\frac{1}{x_n} < 2$

we apply Theorem 2 to obtain

$$\frac{1}{x_n}(x_n - 1) \to 0$$

$$1 - \frac{1}{x_n} \to 0$$

$$\frac{1}{x_n} - 1 \to 0 \qquad \text{[Cor.(B)]}$$

$$\frac{1}{x_n} \to 1 . \qquad \text{[Def.]}$$

(5) $y_n \to b \ (b \neq 0) \Longrightarrow \frac{1}{y_n} \to \frac{1}{b} .$

Proof: $\frac{y_n}{b} \to 1$ [2]

$\frac{b}{y_n} \to 1$ [4]

$\frac{1}{y_n} \to \frac{1}{b}$ [2]

(6) $x_n \to a$, $y_n \to b$ $(b \neq 0) \Longrightarrow \frac{x_n}{y_n} \to \frac{a}{b}$.

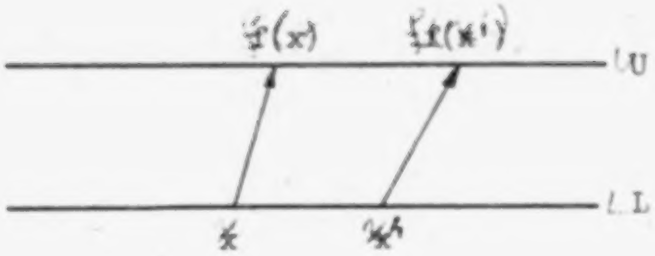
Proof: $x_n \to a$

$\frac{1}{y_n} \to \frac{1}{b}$ [5]

$x_n \cdot \frac{1}{y_n} \to a \cdot \frac{1}{b}$. [3]

4. The Derivative as Local Multiplier.

Consider a function $f: x \to f(x)$ as a mapping from a lower line L to a parallel upper line U .



Then $f'(x)$, the derivative at x ,

$$\text{the } \lim_{x' \to x} \frac{f(x') - f(x)}{x' - x} ,$$

may be interpreted as a local multiplier or multiplier at x .

Example 1. $f: x \to ax + b$

$f(x') - f(x) = a(x'-x)$

In going from L to U the step $x'-x$ is multiplied by the constant a .

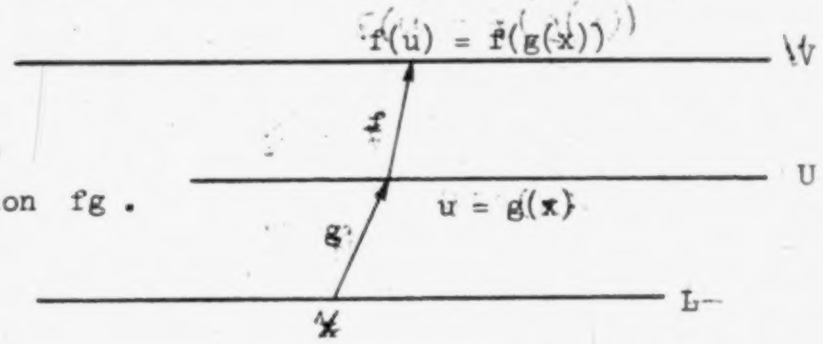
Example 2. $f: x \to x^2$

$f(x') - f(x) = x'^2 - x^2 = (x'+x)(x'-x)$

$x+x'$ is the multiplier. With x fixed we let $x' \to x$. The limiting multiplier or multiplier at x is $2x$.

This interpretation makes the chain rule very easy to understand.

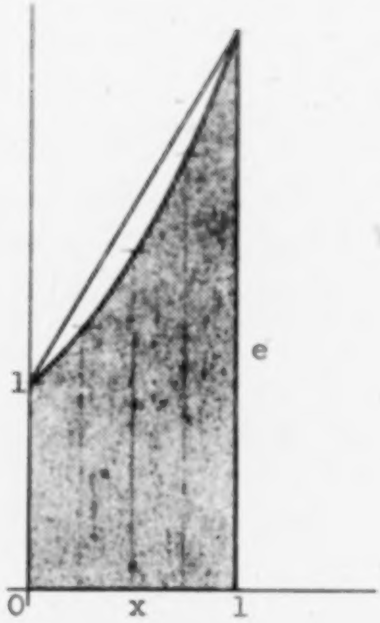
If $g: x \to g(x)$
and $f: u \to f(u)$
$x \to f(g(x))$ defines the composite function fg .



As we go from L to U, the multiplier at x is $g'(x)$. From U to V the multiplier is $f'(u) = f'(g(x))$. The result of these two multiplications is $f'(g(x)) \cdot g'(x)$.

Polynomial Approximations for e^x, $\sin x$, $\cos x$

Since $De^x = e^x > 0$
e^x increases. Hence on the
interval $0 < x < 1$
$1 < e^x < e$.
The area above $[0,x]$ is $e^x - 1$.
With $x = 1$ we obtain $e - 1$ which
is less than the area of the
trapezoid $(D^2e^x > 0)$.



Then

$$e - 1 < \frac{1+e}{2}$$

whence $e < 3$.

So we write

$$1 < e^x < 3 \quad \text{on} \quad (0,1].$$

Integrating from 0 to x

$$x < e^x - 1 < 3x.$$

Repeating

$$\frac{x^2}{2} < e^x - 1 - x < 3\,\frac{x^2}{2}$$

$$\frac{x^3}{2\cdot 3} < e^x - 1 - x - \frac{x^2}{2} < 3\,\frac{x^3}{2\cdot 3}.$$

Adding $1 + x + \frac{x^2}{2}$ throughout

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} < e^x < 1 + x + \frac{x^2}{2!} + 3\,\frac{x^3}{3!}.$$

Better approximations are easily obtained.

Similarly from

$$0 < \cos x < 1 \qquad (0 < x < \tfrac{\pi}{2}.)$$

we obtain by repeated integration

$$0 < \sin x < x \ ,$$

$$0 < 1\text{-}\cos x < \frac{x^2}{2} \ ,$$

$$0 < x\text{-}\sin x < \frac{x^3}{3!} \ ,$$

$$0 < \frac{x^2}{2} + \cos x - 1 < \frac{x^4}{4!}$$

from which it follows that

$$x - \frac{x^3}{3!} < \sin x < x$$

$$1 - \frac{x^2}{x!} < \cos x < 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \ .$$

Better approximations are easily found.

A Proposed Elementary Functions/Calculus Chapter 0

Purpose.

To agitate but also stimulate; to uncover a deficiency of mathematical power; to present a global view of a particular area of mathematics; to allude to the underlying unities of this area of mathematics; to establish a set of goalposts for the year's work; NOT to arrive at any particular numerical solutions; NOT to develop any vocabulary of the calculus.

Rationale.

Move from a short opening statement to a free-wheeling set of problem/query situations where the emphasis is entirely on "how could we get at ..." and "what are the mathematical processes needed in ..." without trying to complete the solution.

Warning.

This material is meant to be nonstructured and very open-ended. Throw out your conventional expectations of competence and achievement of full understanding and even every student getting somewhat to the same point of understanding.

Students Text - Chapter 0

You are asked in this chapter to break out of your traditional thinking and approaches in what might appear to be many different directions. To be candid, there is an implicit unified body of ideas but do not actively seek them -- let the "big picture" sneak up on you. What will be explicit is a series of wide-ranging situations-with-a-query. Each will be followed by some directive discussion or questions to help you wrestle with all of the aspects of the situation.

But play the game straight: first restrict yourself to the original statement alone and try to gain a real depth of understanding of the inherent problems as well as some ideas of how one might go about solution. Seek a mathematical formulation, an awareness of potential pitfalls, and methods of attack -- not answers. (The actual power to handle these situations and arrive at a numerical answer may or may not evolve over the year but that is singularly unimportant at this point.)

After an honest initial attempt and some discussion with other students, read on and see how complete your thinking was. You may find that you understood more than was hope for or you may find some factors that you missed.

One of the vital aspects of this chapter is the verbalization of the ideas. Self-understanding of the situations is not sufficient: you must gain the ability to discuss the ideas in a manner that will reach others -- and this is often far more difficult than you might guess!

Think freely -- think alone -- think together -- think!!

Problem 1.

Eighth graders frequently become enamored with patterns and long strings of numbers. One child presented this "number" to his teacher:

$$\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots$$

He wanted to know, "Is it a number? If so, what number?"

His teacher wasn't much help but did comment cryptically that decimals might be a help and multiplying it by 4 might help him recognize an old friend.

Some Suggested Directions of Thinking for Problem 1.

(1) What does "..." do to this situation? Is the sum bounded or unbounded? Explain carefully why you feel this without reference to any partial sums or any "cutting off" of the "string".

(2) Take some partial sums and form a table $(S_1 , S_2 , S_3 , \dots ,$ e.g.,

$S_3 = \frac{1}{1} - \frac{1}{3} + \frac{1}{5}$). Start with the first term and work out slowly -- share the arithmetic burden and don't be overzealous in terms of quantity or accuracy. Explain what is happening. Do decimals help?

(3) Did you consider plotting n against S_n to get a graphical representation of the partial sums? What is the physical term associated with this type of behavior?

(4) If a computer gave you S_{50} (the sum of the first fifty terms), could you make any productions about is size? Or about the size of S_{51} ?

(5) If a computer presented you with S_{72} , could you quickly give an expression of S_{73} ? Or, in general terms, given S_n , can you produce S_{n+1} ?

(6) Can you make any cogent argument that answers the child's first question affirmatively? If so, does your argument say what kind of a number it is? (i.e., positive negative, integral, rational, irrational,???)

(7) What does multiplying it by 4 do? Do the results of multiplying by 4 actually change the nature of the number at all?

(8) Do you have any answer for the child's second question -- wild guess or supportable? (This question is the least important aspect of this problem if you refer to the opening remarks of this chapter!)

Problem 2.

Every eighth grader (see Problem 1) has a friend, so . . .

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots$$

Again, the child asked, "Is it a number? If so, what number?" However, this time the teacher suggested far more caution.

Some Suggested Directions for Thinking in Problem 2.

(1) Forming partial sums again is an obvious approach. Don't over-labor the arithmetic but the note of caution argues for going rather far out "the string". Share the work and you might find that decimals are easier to work with for approximation.

(2) What does a graphical representation show this time?

(3) Take a few moments out to consider a "number" formed by adding one, one $\frac{1}{2}$, two $\frac{1}{4}$'s, four $\frac{1}{8}$'s, eight $\frac{1}{16}$'s, sixteen $\frac{1}{32}$'s and so on.

$$\frac{1}{1} + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}} + \underbrace{\frac{1}{16} + \frac{1}{16}} + \cdots$$

By considering the grouped terms, one can make an interesting conclusion about the size of this "number".

(4) Make a convincing argument that the "number" in 3 is more than three. More than five and one-half. More than ten. More than any given positive integer.

(5) Make a term-wise comparison of the "number" in 3 and the given problem. That is, consider the order relation (who's bigger) of the corresponding terms (e.g., compare the two fifth terms or compare the two tenth terms or compare the ninety-ninths).

(6) What order relation does this imply about the two "numbers"? What does this imply about the child's question?

(7) Try to make a convincing argument that the given "number" is greater than three. Can you generalize your argument for any given positive integer?

(8) What change would alternating the signs make? That is, glance briefly at

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

Problem 3.

Consider

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$$

in the framework of the preceding problems.

Note:

The data for Problem 4 may not be realistic -- given a drag strip magazine, it could be corrected.

Problem 4.

Using a high speed camera with a synchronized timing device, the officials at the local drag strip produced the following table of time elapsed relative to nearest distance post for the winning car.

time (sec)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
distance post (feet)	0	10	20	50	90	140	200	280	370	470	580	700	840	990	1150	1320

It follows immediately that this driver did the quarter mile in 15 sec. so his average speed was 88 ft./sec. or 1 mile/min. or 60 mile/hour. But the interesting question is his actual speed when he crossed the finish line. Having accelerated from a standing start (0 mph), what final speed enabled him to average 60 mph ? For example, his average speed on the last half of the quarter mile was . . .

Some Suggested Directions of Thinking for Problem 4.

(1) One approach to this situation is through considering the average speed for the last 10 , 9 , 8 , ... seconds of travel. Say S_{10} , S_9 , and so on. If you have not worked with these, it may get you started. Form a table to refer to. (Working jointly will share the labor and minimize the arithmetic errors.)

(2) If S_n is the average rate of speed for the n seconds preceding the finish line, what is the domain of n with the given data? What is the S_n that you are looking for?

(3) For any two S_n's , say S_a and S_b where $a > b$, what is the order relation? (That is, can you be certain which is bigger?)

(4) For any three S_n's , say S_a , S_b , S_c where $a > b > c$, how is S_b related to S_a and S_c ? Can you be sure of which one S_b is closest to?

(5) The word "nearest" in the original statement of the problem has what effect on the data?

(6) Suppose the officials added the information that at $14\frac{1}{2}$ seconds the nearest distance post was the 1220 . What does this seem to imply?

Problem 5.

Consider a certain virulent bacteria that manages to reproduce by mitosis (cell division) once a minute. That is, starting with one bacterium after one minute there will be two, after two minutes there will be four, after three minutes there will be eight, and so on. Given a zero mortality factor, what is the rate of change of the number of germs after 10 minutes?

Some Suggested Direction of Thinking for Problem [illegible].

(1) The average rate of change can be expressed as

$$\frac{N-n}{T-t} \text{ germs/sec.}$$

e.g., for the first 10 seconds

$$N = 2^{10} \ , \ T = 10$$

$$n = 2^{0} \ , \ t = 0$$

so $$\frac{2^{10}-2^{0}}{10-0} = \frac{1024-1}{10} = 102.3 \text{ germs/sec.}$$

or an average rate of change of about 102 germs per second.

But for the next 10 seconds

$$N = 2^{20} \ , \ n = 2^{10} \ , \ T = 20 \ , \ t = 10$$

so $$\frac{2^{20}-2^{10}}{20-10} = \frac{2^{10}(1023)}{10} = 2^{10}\,(102.3) \text{ germs/sec.}$$

or an average rate of change of about 12,600 germs per second.

Stop and consider these ideas before reading on and see if it opens a new approach.

(2) The results found in (1) might be thought of as C_{-10} (average rate of change for preceding 10 sec.) and C_{+10} (average rate of change for succeeding 10 sec.). Give some thought to C_n where n is an integer > -10 but $< +10$. Can $n = 0$?

(3) Form a table of all C_n's where $|n| \leq 10$ and n is an integer. Work together to get the computation out of the way quickly and then consider the results carefully.

(4) Consider your C_n's. Form a careful argument why "C_0" is not the average of C_{-1} and C_1 or of C_{-2} and C_2, etc.

(5) Will "C_0" be closer to C_{-1} or C_{+1} ? Why?

(6) How does this whole vein of thought compare to the Problem 4? Reread

the suggested lines of thought for Problem 4 and see if they can be used on your work in this problem.

Problem 6.

Infinite patterned (but non-repeating) decimals can be added to get some interesting results. Consider this problem:

.1 3 5 7 9 1 1 1 3 1 5 1 7 1 9 2 1 2 3 2 5 2 7 2 9 3 1 - - -
+ .2 4 6 8 1 0 1 2 1 4 1 6 1 8 2 0 2 2 2 4 2 6 2 8 3 0 3 - - -

Some Suggested Directions for Thinking in Problem 6.

(1) What kind of numbers are these? That is, are they rational or irrational? Do you expect the sum to be rational or irrational?

(2) With infinite decimals, the actual summing can be done from left to right with the "carry" handled afterward:

Example:

.2 9 2 9 2 9 2 9 2 9 - - -
+ .3 4 5 6 3 4 5 6 3 4 - - -
.5(13) 7(15) 5(13) 7(15) 5(13)- - -

which can be rewritten as

.6 3 8 5 6 3 8 5 6 3 - - - or $.\overline{6385}$

This should help you add the given numbers correctly!

(3) Is your answer a repeating decimal? Can it be written with an indicated repeating block?

(4) If the suspected repetition ever breaks down, where might it first happen? Is it enough that if the repetition continues for 48 places or 96 places it will continue forever?

(5) One tricky spot occurs when the "numbers" in the pattern become three

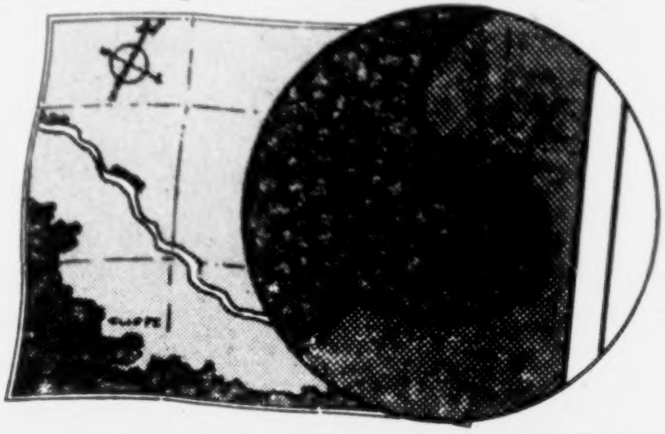
digit numbers. Specifically, consider this segment of the given problem:

```
   - - - 8 9 9 1 9 3 9 5 9 7 9 9 1 0 1 1 0 3 1 0 5 1 0 7 - - -
 + - - - 8 9 0 9 2 9 4 9 6 9 8 1 0 0 1 0 2 1 0 4 1 0 6 1 - - -
 ______________________________________________________________
```

(6) Discuss this statement relative to this problem: "If a pattern continues for n steps, it continues forever."

Problem 7.

Below is a careful field map drawing of a high cliff and the known position of a sniper in a cave (x) .



He is pinned down from above and cannot lean out without being shot. Your problem, as leader of the troops, is to delineate his field of fire and avoid it.

(Note: there is no absolute need for numbers in this problem -- do not coordinatize; just use the picture.)

STOP

Some Suggested Directions for Thinking in Problem 7.

(1) Did you think of his fire field as a set of rays? Geometrically, what is covered by this set? That is, is his fire field the interior of an angle, the exterior of an angle or a half-plane?

(2) To you and your troops, which is more important, the interior of the fire field or the boundaries of his fire field?

(3) The boundary of his fire field is related to the cliff in a special manner. Discuss this and evolve a name for this situation -- rely on your present vocabulary. Are the boundaries "constructible"?

(4) How is the boundary of the sniper's fire field influenced by the cliff or by his position on the cliff? That is, how would your solution change if he was moved 50 yards west or 25 yards north?

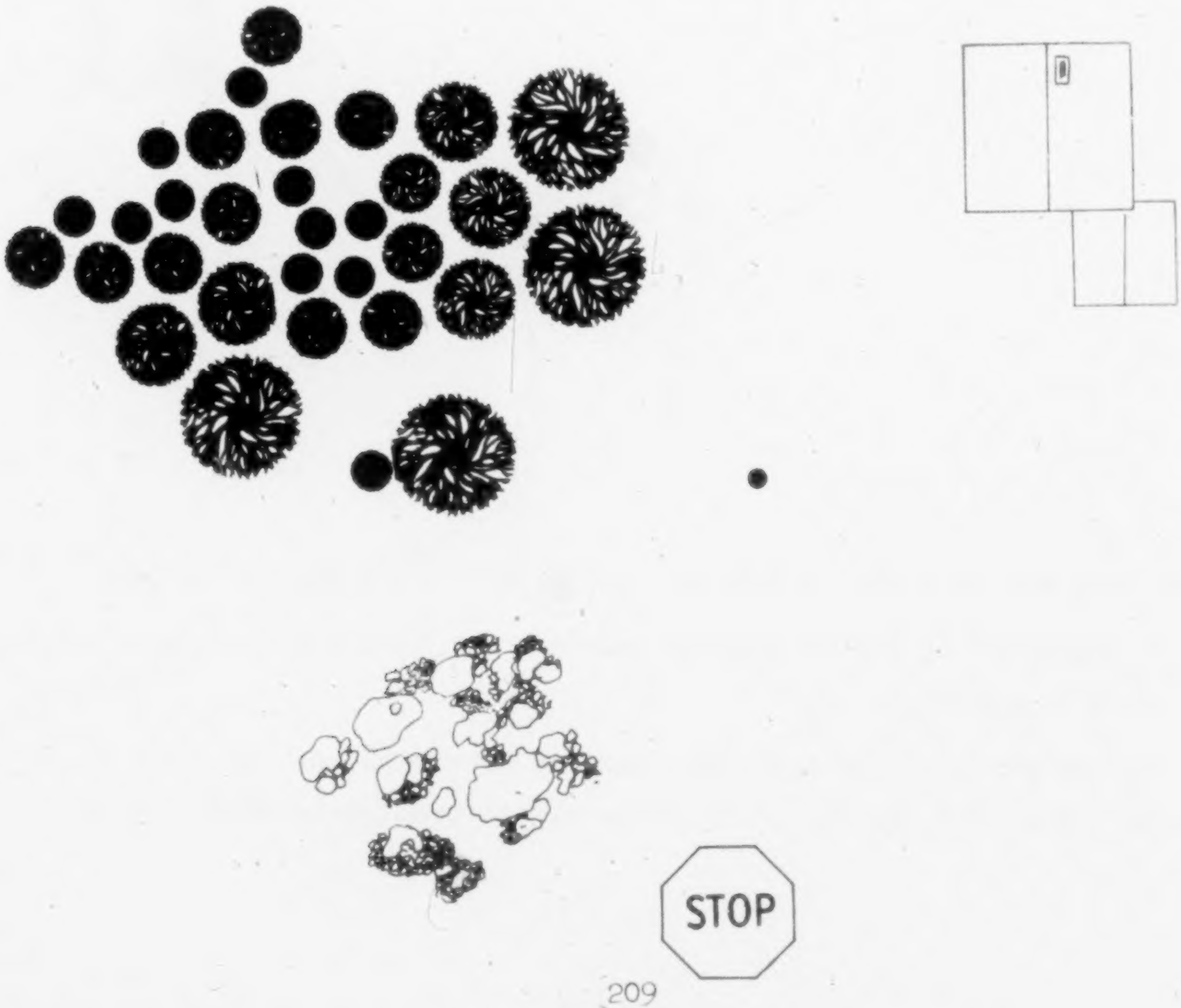
(5) If the sniper had complete freedom of choice for his position on the cliff, are some positions more advantageous for his purposes? less advantageous?

(6) Theoretically, how close could you approach his position without being fired upon?

Problem 8.

The Park Police have a permanent watchman's post at the point x . There are numerous dense woods, boulders and buildings in the vicinity as shown. Determine the watchman's field of vision so you can enjoy the park without detection.

(Note: again use the picture without introducing numbers.)



Some Suggested Directions for Thinking in Problem 8.

(1) Again, the field of vision is essentially a set of rays but some of them are now line segments. Did you try to show geometrically what his field of vision is? Can you construct the boundaries?

(2) Can you geometrically describe the "safe areas"?

(3) How are the boundaries of the field of vision related to the obstacles? Discuss this using an acceptable amount of mathematical words.

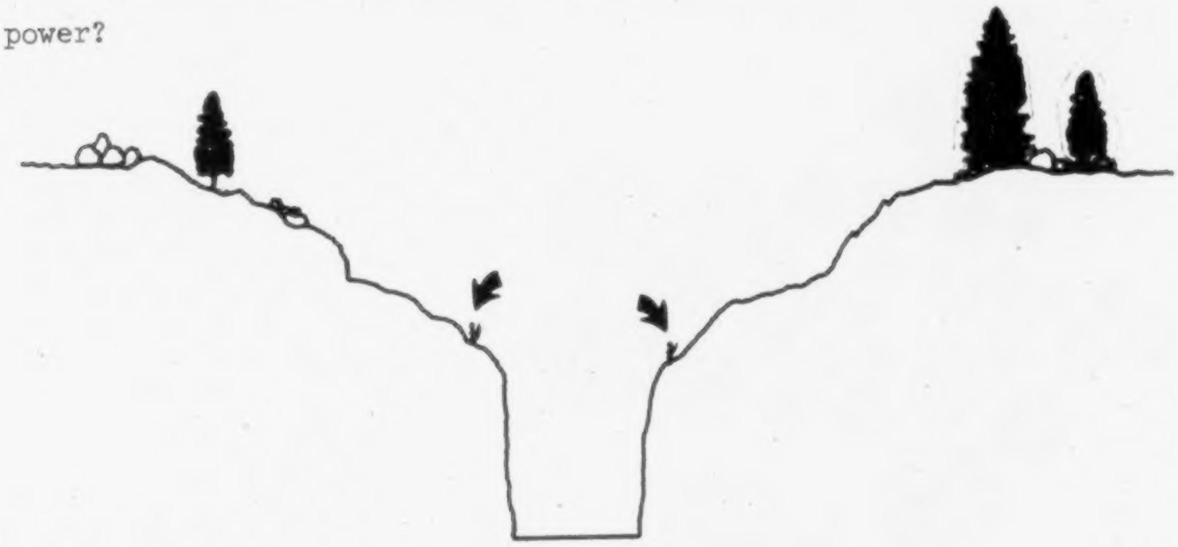
(4) What is the essential difference between this problem and problem 7? Does it remind you of "at a point on a ..." and "from a point to a ..."?

(5) Could the watchman choose a better position for his post? That is, how does changing his position affect the "safe areas"?

Aside.

Another interesting twist:

The Indians are positioned at the indicated points on a mountain pass. Is it possible to sneak through without being under dangerous fire power?



Other problems with like potential that haven't been written up yet:

9. A summation of finite moments leading to infinite summation -- such as work or pressure.

10. An extremum problem such as a maximum area with one free wall -- must it be completely restricted to rectangles? quadrilaterals? polygons?

11. An area problem by successively smaller segments -- perhaps a kidney-shaped swimming pool.

12. A volume problem by successively thinner slices -- perhaps a vase or a headlight.

Final Comments.

1. Probably a maximum of 10 good problems should be used. The ordering of them should be reconsidered since the present ordering is how they popped out in installments.

2. A carefully developed teacher's commentary might be necessary. It might point out, for the teacher only, the underlying intents of each problem. Also it might indicate some methods of generating and maintaining the discussion and involvement. Parenthetically and pessimistically, if the teacher needs this commentary, the chapter will probably be a bust anyway.

3. The chapter should probably entail a maximum of two weeks work and the teacher should be encouraged to cut it off short rather than have it die on the vine. Under-development of the ideas is far less risky than the "all orifices" approach -- it requires some sensitive and sensible pedagogical judgment as well as some mathematical competence . . . a bad pair of qualities to rely on?????

A Naive Theory of Integration

How do you count the change in your pocket? Do you draw out each coin in turn and add the amount of each new coin to the preceding total:

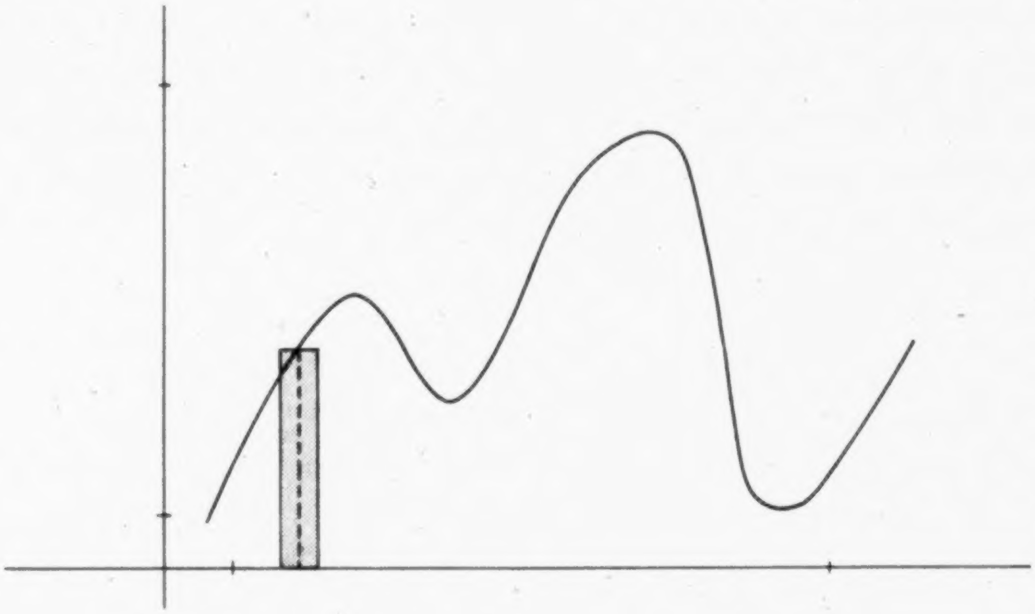
$$5 + 1 + 10 + 1 + 1 + 10 + 25 + 1 + 5 + 10 + 1 + 10 + 1 \ ?$$

Or do you dump all the coins out on the table and find that you have 1 quarter, 4 dimes, 2 nickels, and 6 pennies and thus that you have

$$25 + 40 + 10 + 6 = 81 \text{ cents?}$$

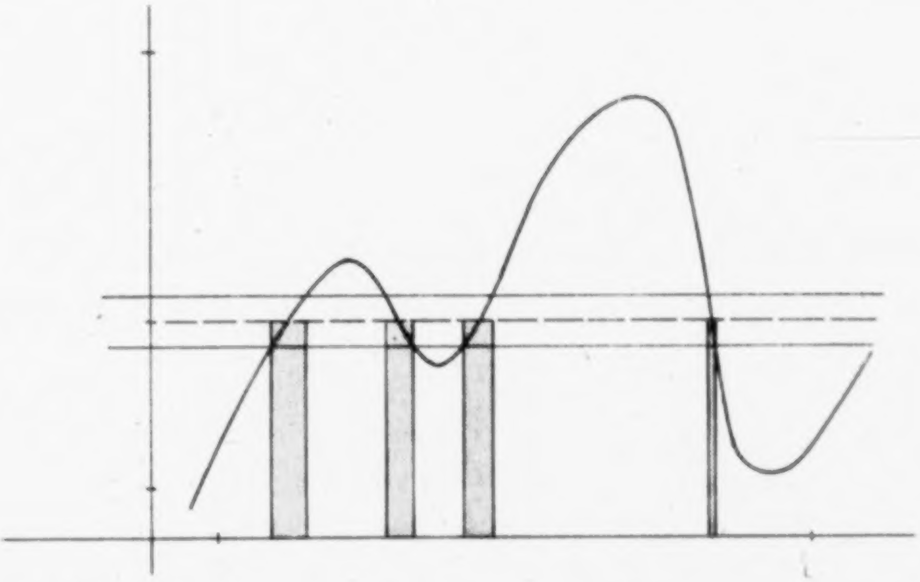
The latter method is certainly efficient and it is in fact the one used in many problems and techniques from probability.

Much the same issue arises in defining the area under a curve. Suppose that a function is given from the interval $[a,b]$ into the interval $[c,d]$.



Traditionally (Riemann integral) we always subdivide the x-axis into intervals of width, say $(b-a)/n$, and thus add up, sequentially from left to right, a series of rectangles of this fixed width and of varying heights, $f(s)$.

Instead, why not subdivide the y-axis into intervals of length $(d-c)/n$?



Then we should add up a set of rectangles of fixed height t but with varying widths. We can think of this as one "rectangle" of height t whose base is the measure of the set $\{x: |f(x) - t| < 1/n\}$.

This second approach is that traditionally taken in the development of the Lebesgue integral. Could such an approach to integration be taken in high school? It would seem that it is at least as intuitive as the classical Riemann integral. It does have the advantage of following naturally from procedures used in probability and statistics. It puts the technical difficulties into assigning measures (length) to subsets of the real line, rather than into measure (area) in the plane.

A serious attempt to develop this approach might show that it has fewer difficulties than the classical one. The power of the Lebesgue integral is well known -- as well as its computational weakness. While I would doubt if all the subtleties would be apparent at Grade 12, the methods might. I do actually believe that we will see the day when the pleas from the physicists and engineers for the Lebesgue treatment of Fourier series and integral transforms will place such a development early in the college curriculum. If I am close to being correct then this "Lebesgue" method would build important readiness for college.

Here are some of the group's reaction to keep in mind:

1. The Lebesgue integral deals most naturally with a definite integral of a

bounded function. It may be less natural as an indefinite integral. (This could be a blessing, since it would force us to write $\int_a^x f$ instead of $\int f$.

2. To evaluate $\int f$ you need an anti-derivative. Wouldn't you thus need the Riemann integral too?

3. Lebesgue integral is natural for problems like moment of inertia problems.

4. One member of the group adds a plea for both treatments. He endorses the Lebesgue integral for the reasons above, putting most weight on the mathematical power. He notes that the Riemann integral is most natural for problems involving growth. As a mathematical application of a growth problem he cites the Cauchy Integral Test. Also $\int x^2$ is easier for R-integral since an even subdivision of the y-axis is analogous to integrating $\int \sqrt{x}$ in the R-integral case.

APPENDIX A

Geometry Problems for the Grades 10-12 Block

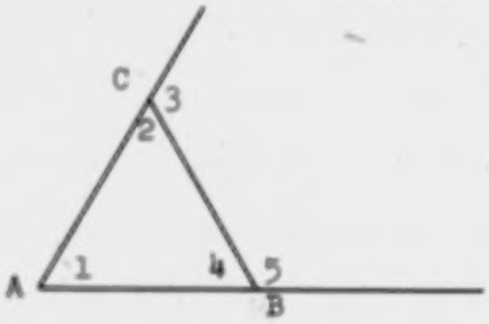
These problems are, of course, available for any appropriate position in the Grades 7-12 sequence. However, they were developed with the Synthetic Geometry Block for possibly Grade 10 in mind.

It has been suggested several times that students should develop the ability to analyze a "real life" problem situation, from a mathematical model (see various papers on modeling) and come to a reasonable understanding, appreciation, and solution through the relationships between the situation and the model. It was felt that both writers and teachers should have available a collection or "kit" of such problem situations, which are not always easily invented. This set of problems give a good many such suggestions in more less detail, and are often open-ended. A point of view that should not be neglected is puzzles and problems, "just for fun". A few of these are included in this appendix.

Coordinate-free Geometry

1. (After: isosceles triangles and angle sum in a triangle)

Given $\triangle ABC$, with $\overline{AC} \cong \overline{BC}$



(1) If $m\angle 1 = 70$, find m of the other angles.

(2) If $m\angle 3 = 160$, find m of the other angles.

(3) If m $\angle 5 = 105$, find m of the other angles.

(4) If m $\angle 1 = x$, express m of the other angles i.t.o. x .

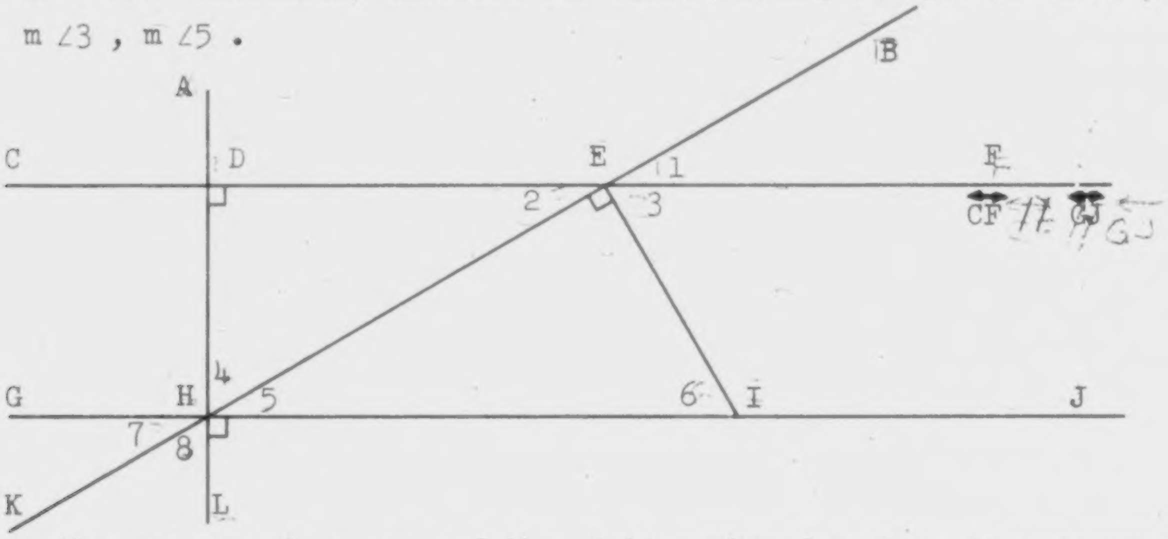
(5) For what m $\angle 1$ will m $\angle 3 =$ m $\angle 5$?

(6) For what m $\angle 1$ will m $\angle 3 <$ m $\angle 5$?

(7) What are the domains of m $\angle 1$, m $\angle 2$, m $\angle 3$?

(8) As m $\angle 1$ increases, describe the corresponding changes in m $\angle 2$, m $\angle 3$, m $\angle 5$.

2.



$\overleftrightarrow{CF} \parallel \overleftrightarrow{GJ}$

(Given the measure of any one of the eight numbered angles, the measures of the others can be determined.)

(1) If m $\angle 1 = 40$, find the measure of the others.

(2) If m $\angle 6 = x$, express the m of the others i.t.o. x .

(3) If m $\angle 3 = 2$ m $\angle 1$, find m of the others.

(4) If m $\angle 5 <$ m $\angle 4$, relate the m of each pair of the others. (28 such)

(5) If $\overrightarrow{HB}$ rotates counterclockwise about H

  (a) Describe the changes in all the numbered angles.

  (b) Describe the changes in the segments $\overline{DE}$, $\overline{EH}$, $\overline{EI}$, $\overline{HI}$.

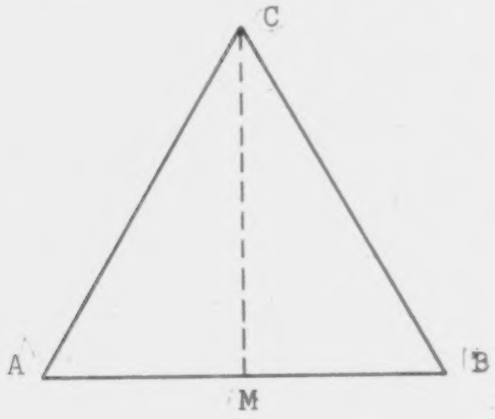
  (c) Describe the changes in the ratios:

$$\frac{DE}{DH} , \frac{DE}{EH} , \frac{DE}{HI} , \frac{EI}{EH} , \frac{EI}{HI} , \frac{EI}{DE} .$$

3. Construction Problems.

Students should be able to discuss the possibility or impossibility of these constructions.

(1)

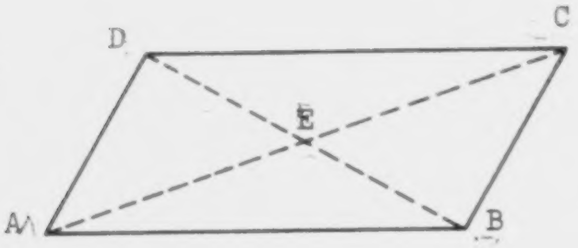


(a) Construct $\overline{CM}$ to bisect $\angle ACB$, making it perpendicular to $\overleftrightarrow{AB}$.

(b) If M is the midpoint of $\overline{AB}$, draw $\overline{CM}$ to bisect $\angle ACB$.

(c) If M is the midpoint of $\overline{AB}$, draw $\overline{CM}$ perpendicular to $\overleftrightarrow{AB}$.

(2)



(a) Connect A and C to bisect $\overline{DB}$.

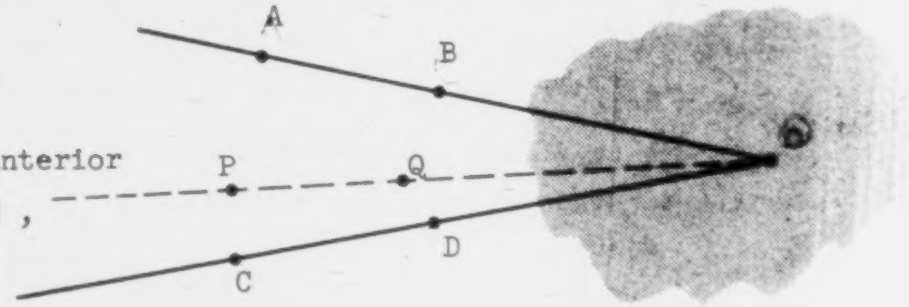
(b) If E is the midpoint of $\overline{AC}$, draw line $\overleftrightarrow{DEB}$.

4. Elementary ("familiar") Construction Problems with Constraints.

(1) Given $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$ with inaccessible intersection O

(a) Bisect $\angle AOC$

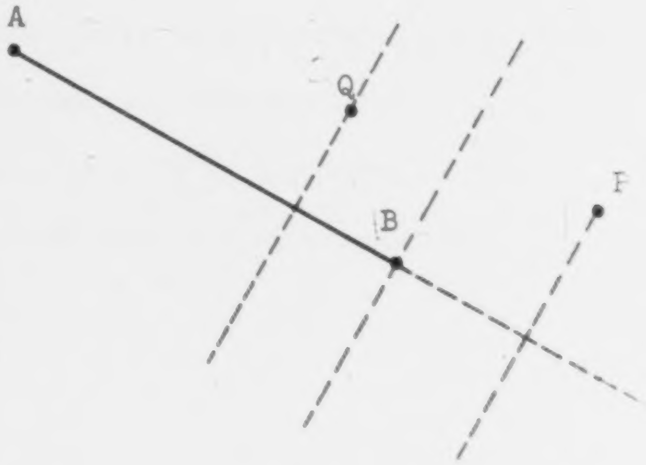
(b) If P is an interior point of $\angle AOC$, construct $\overleftrightarrow{PQ}$ through O.



(2) Given segment $\overline{AB}$

(a) Construct a perpendicular to $\overleftrightarrow{AB}$ at B, without extending $\overline{AB}$.

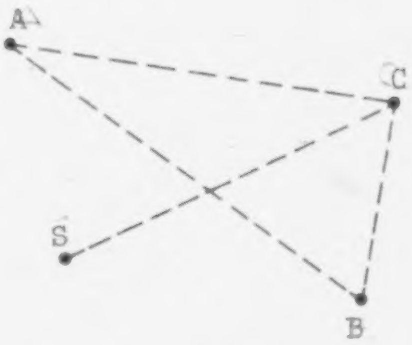
(b) Given point P "beyond" $\overline{AB}$, construct $\overleftrightarrow{PR}$ perpendicular to $\overleftrightarrow{AB}$ without extending $\overline{AB}$.



(c) Given point Q off $\overleftrightarrow{AB}$, construct $\overleftrightarrow{QS}$ perpendicular to $\overline{AB}$ without using any point of AB as center of a circle.

5. Constructions with Constraints.

(Compasses only, Mascheroni constructions)



(1) Given "segment" $\overline{AB}$, double it; triple it.

(2) Find any number of points on $\overleftrightarrow{AB}$.

(3) Bisect $\overline{AB}$.

(4) Given "triangle" ABC , construct $\overleftrightarrow{CP} // \overleftrightarrow{AB}$.

(5) In "triangle" ABC , construct $\overleftrightarrow{CQ} \perp \overleftrightarrow{AB}$.

(6) Construct $\overleftrightarrow{BR} \perp \overleftrightarrow{AB}$.

(7) Find the intersection of $\overline{AB}$, $\overline{CS}$.

(8) Find the fourth proportional to $\overline{AB}$, $\overline{CS}$.

(9) Construct square ABXY .

6. Constructions with Constraints.

(Euclidean compasses, "snap compasses". "A circle can be drawn with a given center and given radius." This is not a justification for "On $\overleftrightarrow{AB}$ mark P , Q , so $\overline{PQ} \cong$ given segment". Euclidean compasses can be used to draw a circle with given center and radius, but cannot "move" that radius. As soon as you lift the compasses they snap closed and you lose the setting.)

(1) Double (triple) a given segment $\overline{AB}$.

(2) Bisect a given segment, a given angle.

(3) Construct a perpendicular to a given line (2 cases) .

(4) Add (subtract) two given segments.

(5) Through a given point construct a line parallel to a given line.

(6) Construct a fourth proportional to three given segments.

7. Three-point Constructions.

A triangle is determined by its three vertices, A , B , C , given in position; then also we can find, e.g., the midpoint of its sides, M_a , M_b , M_c . If we now remove all but points A , B , M_b , leaving them in position, we can still find the unique original triangle ABC . In these problems we are given three points in position and are asked to find the original $\triangle ABC$.

(1) A B M_c (Redundant)

(2) A B M_b

(3) A M_a M_b

(4) A M_b M_c

(5) M_a M_b M_c

(6) A B O (circumcenter) (Not determined - why?)

(7) A B I (incenter)

(8) A O M_a

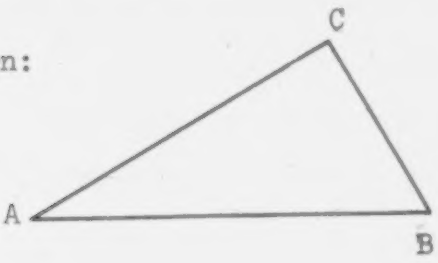
(9) A O M_b (Needs discussion)

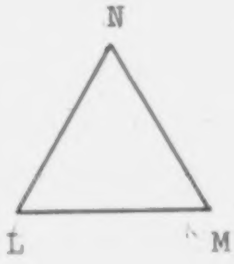
(10) O M_a M_b

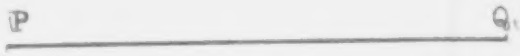
(There are many more difficult and interesting problems here.)

8. Construction Problems (Areas).

Given:







(1) Construct isosceles $\triangle ABD \approx \triangle ABC$, with $\overline{AD} \cong \overline{BD}$.

(2) Construct isosceles $\triangle ABE \approx \triangle ABC$, with $\overline{AB} \cong \overline{AE}$.

(3) Construct right $\triangle ABF \approx \triangle ABC$, with right angle at B .

(4) Construct right $\triangle ABG \approx \triangle ABC$, with right angle at G .

(5) Construct isosceles right $\triangle ABH \approx \triangle ABC$ (*Impossible?)

(6) Construct equilateral $\triangle ABJ \approx \triangle ABC$ (*Impossible?)

(7) Construct an isosceles right $\triangle \approx \triangle ABC$ (Compare with (5)).

(8) Construct an equilateral $\triangle \approx \triangle ABC$ (Compare with (6)).

(9) Construct $\triangle PQR \approx \triangle ABC$.

(10) Construct right $\triangle PQS \approx \triangle ABC$, with right angle at Q . (Students should see that this is a combination of (9) and (3).)

(11) Construct an isosceles $\triangle PQT \approx \triangle ABC$, with base $\overline{PQ}$. (Students should see that this is a combination of (9) and (1).)

(12) Construct rectangle $ABKZ \approx \triangle ABC$.

(13) Construct rectangle $PQXY \approx \triangle ABC$. (Students should see that this is a combination of (12) and (9).)

(14) Construct a square $\approx$ a given rectangle.

(15) Construct a square $\approx \triangle ABC$ (Combine (12), (14)).

(16) Construct $\triangle A'B'C' \sim \triangle ABC$ and $\approx \triangle LMN$.

(17) Construct a single $\triangle \approx$ the sum of the areas of $\triangle ABC$ and $\triangle LMN$.

(18) Construct a single $\triangle \approx$ the difference of the areas of $\triangle ABC$ and $\triangle LMN$. (Students should be able to extend (17), (18) to find a triangle whose area is a linear combination of the areas of $\triangle ABC$ and $\triangle LMN$, e.g., $3(\triangle ABC) - 2(\triangle LMN)$).

(19) Construct a square $\approx$ the sum of the areas of $\triangle ABC$ and $\triangle LMN$. (Students should see that this is a combination of (17) and (14).)

(20) Construct $\triangle A'B'C' \sim \triangle ABC$ and $\approx \triangle LMN$ (as in (16)). Then construct $\triangle A''B''C'' \sim \triangle ABC$ and $\approx$ the sum of the areas of $\triangle ABC$ and $\triangle LMN$. (Students should be led to apply the Pythagorean theorem here.)

(21) Construct a rectangle on a given base, $\approx$ a given rectangle.

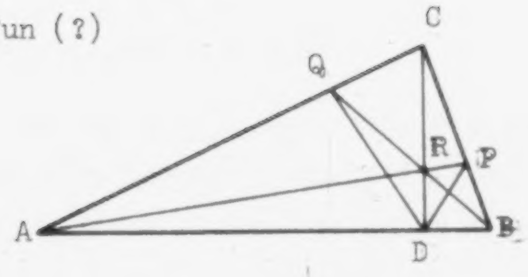
(22) Construct a rectangle $\approx$ the sum of the areas of two given rectangles.

(23) Construct a triangle $\approx$ a given polygon.

Students should eventually be able to construct a square $\approx$ to any polygon or combination of polygons, by combining certain key constructions in the collection above -- a natural place for flow chart analysis.

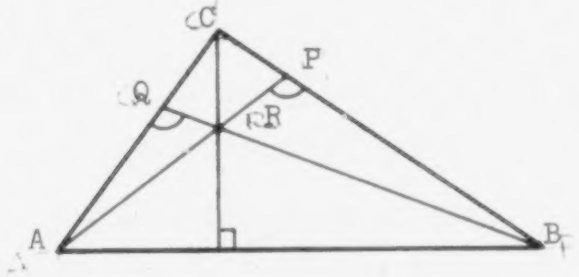
9. Problems.

Just a few miscellaneous ones -- for fun (?)



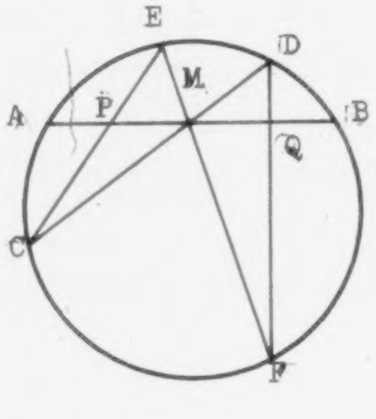
(1) R is any point on altitude CD , and so on.

Prove: $\angle PDC \cong \angle QDC$



(2) R is any point on altitude CD , and so on.

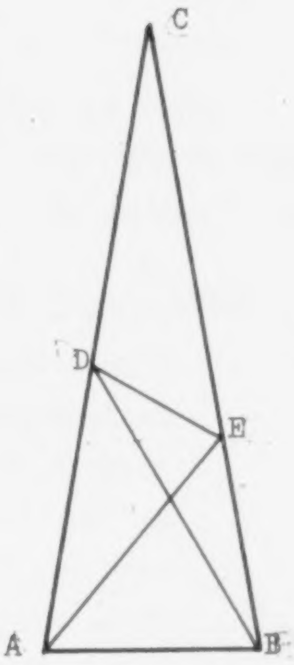
Prove: If $\angle AQB \cong \angle APB$ then they are right angles.



(3) (The butterfly problem)

$\overline{CD}$ and $\overline{EF}$ are two chords through the midpoint, M of $\overline{AB}$, and so on.

Prove: $\overline{PM} \cong \overline{MQ}$.



(4) Given $m\,\angle CAB = m\,\angle CBA = 80$

$m\,\angle EAB = 50$, $m\,\angle DBA = 60$

Prove: $m\,\angle EDB = 30$

or

Find: $m\,\angle EDB$.

(5) If three circles intersect, their pairwise common chords are concurrent.

Problems Situations Leading to Geometric Models

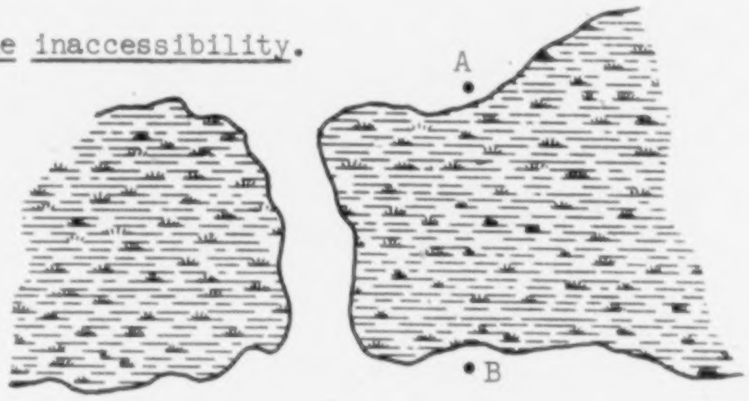
There seems to be a good deal of agreement among the group here with the principle that students should develop the ability to find a mathematical model with which to analyze a situation in the physical world. The phrase "the world of reality" may be an unfortunate one, because it seems to indicate that our mathematical model exists in some other world, which we can reach only by leaving reality. The phrase may be useful with teachers but should not be used with students.

It is excellent pedagogy to have students find or construct these models and then to recall or devise the appropriate mathematical techniques to deal with them. Even if solutions are not found there is value in the search, which can frequently be used to motivate a later and deeper analysis. The detail and rigor should be suitable to the problem and the student -- don't use a micrometer to determine the size of your shoe.

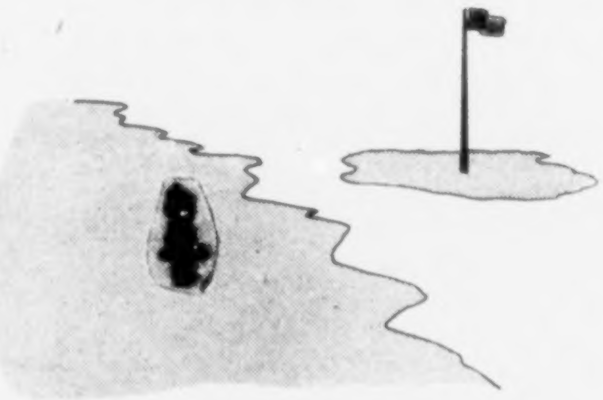
Puzzles and games are fun, but we should be careful not to overemphasize them and leave the idea that they are a significant part of the body of mathematics.

1. Measurements in situations with some inaccessibility.

   (1) In the chapter on Congruence we measured $\overline{AB}$ by finding a point C from which A and B are both visible. Suppose there is no such point?



   (2) Find the height of a flagpole on an inaccessible island. (on top of a building.)



(3) The distance from the earth to the sun is 93,000,000 miles. How was that determined?

(4) Find the size (length) of an egg in a bottle.

(5) How many square inches of skin do you have?

2. Industrial problems.

(1) What is the percent of waste if a maximum circle is cut from a square? (a square from a circle?)

(2) Circular discs are to be punched from a strip of metal of constant width.

  (a) For discs of diameter x what is most economical width?

  (b) What lay-out for strip width y will be least wasteful in production of discs vs. waste, and what is the percentage of waste?

  (c) If the raw stock costs a cents per pound (square inch of uniform thickness), the discs sell at b cents per pound, and the scrap metal at c cents per pound, find the net return for various lay-outs and discuss the effects of small changes in a , b , and c separately and together.

  (d) These questions can be related to corresponding problems in three-space (packing problems).

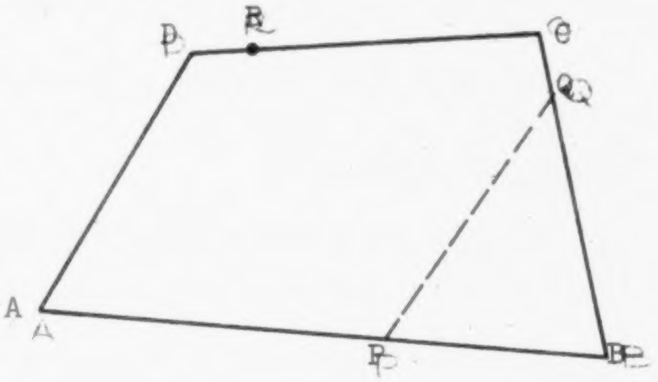
(3) Paper stock comes from the mill in certain standard-size sheets. (20"x30", etc.) It is to be cut into smaller rectangles of specified dimensions with least waste. (Many related problems here: e.g., we may want two different sizes of smaller rectangles in given ration, e.g., 8 (4x5) for every 5 (8x12).)

(4) Assume that 2x4 lumber comes in uniform 12 foot lengths and is to be cut with least waste into x , y , z , length in given ratio, e.g., 6 (5 ft.) for every 4 (8 ft.) for every 3 (9 ft.) for every 8 (3 ft.).

(5) Assume that plastic strips are shipped in 6 foot lengths and are to be cut into x , y , z , lengths in given ratio (as in (4) above), but that the scrap can be sold at half the cost price. Find best (?) production schedule.

3. Less traditional geometry settings.

(1) A triangle is a rigid figure because it is uniquely determined (equivalence class) by SSS , i.e., given SSS we can find the angles and so on. A quadrilateral is not a rigid figure because it is not determined by SSSS . Suppose that ABCD is a quadrilateral whose sides are given.



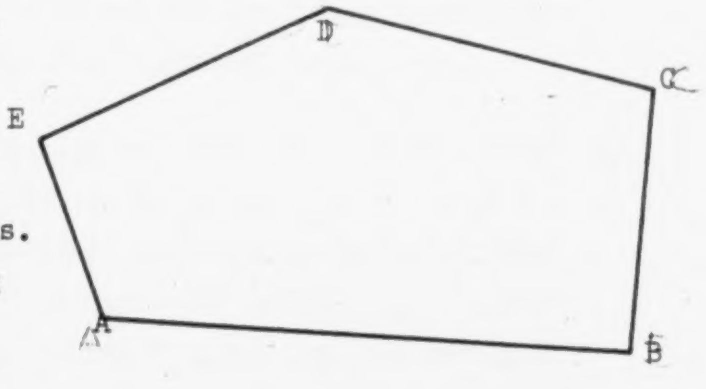
(a) If we join AC by a rigid link does that fix the figure? Discuss.

(b) As above, with P, Q, respectively any points on $\overline{AB}$, $\overline{BC}$? Discuss.

(c) As above, with P, R, any points on a pair of opposite sides? Discuss.

(d) Discuss maximum and minimum size for each angle of the original quadrilateral.

(2) A pentagon with given sides is not determined. Discuss the ways in which we can make the figure rigid. How many new links are sufficient? Generalize the discussion to polygons. Show how diagonal bracing is used to make a box or scaffolding structure rigid. Apply to problem of strengthening a piece of furniture.



(3) Many solid geometry problems are really plane geometry problems in different planes. Students should develop the ability to "see" these planes and apply the proper plane-geometry techniques.

(a) Find radius of the small circle trace of a sphere of radius _r_ which cuts a plane at distance _h_ from its center.

(b) Given the face angles of a trihedral angle, find each of the dihedral angles.

(c) Eratosthenes' measure of the earth (altitude and latitude).

(d) Shortest path across faces of a box (spider and fly).

(e) Show that the lines joining the mid-point of the sides of a skew quadrilateral form a parallelogram.

(f) Find the edge of the smallest cube that will contain two unit spheres.

(g) Find altitude of regular tetrahedron of unit edge.

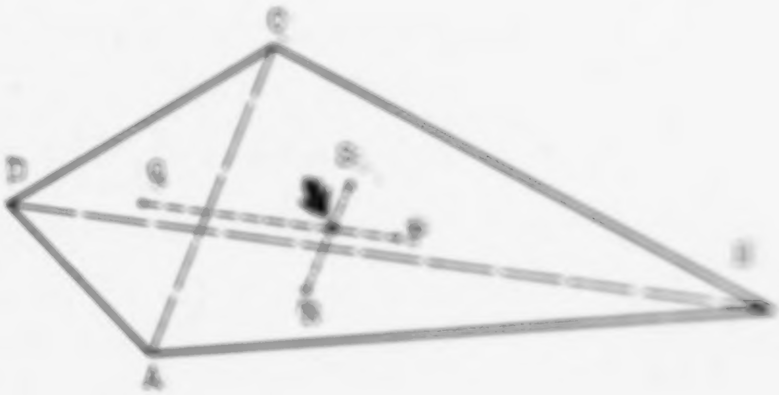
(4) _Puzzles_. Problems with DOMINOES, TETROMINOES, POLYOMINOES. Unicursal curves; (Chromatic) graphs.

Problem Situations (which lead to geometric models)

1. How could you find the inside diameter of a bottle?

2. Before a ship is launched the water line is painted on it. How do they know where to paint it? The topic of floating bodies is excellent for investigation both mathematically and physically. Certainly the "Eureka" story should be told, and students urged to find Archimedes' solution by themselves. Why does a homogeneous wooden prism (cylinder) float horizontally, rather than vertically? Discuss specific gravity as much as needed, and the effects of immersion in fresh water, salt water, air, and so on. It is an interesting experiment to bring in some reasonably regular wooden objects and try to determine (mathematically) where the water line would be, then to test by actual immersion. It is particularly interesting to use several different but similar objects. (If a 6 inch sphere sinks 3 inches will an 8 inch sphere sink 4 inches?) (Puzzle): A rope ladder hanging over the side of a ship has 20 rungs, 1 foot apart. At low tide rung number 4 is at water level. At high tide the water rises 6 feet. Which rung is now at water level?

3. The center of gravity of a figure (polygon) made of a thin homogeneous sheet can frequently be found by mathematical, then physical methods. Show how CG of triangles can lead to CG of a general quadrilateral. (If P is CG of $\triangle ABC$, and Q is CG of $\triangle ABC$, then $\overleftrightarrow{PQ}$ will support ABCD and must contain its CG. Analogously R is CG of $\triangle ABD$ and S is CG of $\triangle CBD$ then $\overleftrightarrow{RS}$ must contain the CG of ABCD, which therefore must be at the intersection of $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$.)



4. What is the longest ladder we can get horizontally into a room? (What is the shape of the room and where are its doors?) This is related to a familiar calculus problem (what is the longest ladder that can turn a corner between perpendicular corridors of widths a and b) which can be extensively generalized.

   4-1. (as above) ... the ceilings are c feet high, and we may tilt the ladders (line segments, rectangles).

   4-2. ... there is immovable furniture with given positions and dimensions.

   4-3. ... we want to get through a maze (orthogonal, general).

   4-4. ... the corridors meet obliquely, at angle α .

   4-5. ... we want to follow a path through a forest of trees with given diameters and positions.

   4-6. Will a given log float all the way down a curving stream?

   4-7. Will a given ship go through a given curved canal?

   4-8. What shape barge of maximum area will go through a given curved canal?

5. Paper folding can lead to a good collection of problem situations involving lengths, areas, volumes, and spatial imagination. They lend themselves readily to mathematical models that lead to predictions which can be tested by measurements.

5-1. Measure a dollar bill, then compute, without further measurement, the length of the crease obtained by folding together a pair of diagonally opposite corners. (The bill is about $2\frac{9}{16} \times 6\frac{3}{16}$, but we may use 3×6 and then generalize.)

5-2. Find the length of the crease if a (unit) square is folded to make one vertex come to the midpoint of an opposite side.

5-3. A (unit) square is folded in half to make a rectangle, which is then folded to make a pair of its opposite corners coincide. If the square is then unfolded, compute the lengths of each segment of the folds and edges, then check by measurement.

5-4. In book-binding a large sheet is usually printed so that after folding and cutting, the pages appear in proper position.

5-4.1 Suppose that an original horizontal rectangle is to be folded LR , BT , LR , to produce a 16 page section. Determine, before the folding, the proper numbering and orientation of the final pages.

5-4.2 As above, but fold BT , LR , LR. (other folds)

5-4.3 As above, but produce a 32 page section by an additional fold (various orders of folding).

5-4.4 After various folds, punch a hole near one corner, then predict the hole positions on each page (after unfolding).

5-4.5 Predict the effects of various cuts of the folded page on the unfolded sheet. (Betsy Ross story: find 1 cut of a folded paper to make a regular star pentagram.)

5-5. Prepare plan (cuts and folds) to form various polyhedra: regular, given irregular (crystal), star.

And still more problem situations

1. If we consider the earth as a sphere, we can devise some significant problems for which there are good mathematical models.

1-1. Find difference in latitude (longitude) between given points A , B . (20N , 30S ; 160E , 150W).

1-2. Find arc length (miles, nautical miles) along given meridian (parallel).

1-3. Find arc length joining any two given points.

1-4. Discuss: bearing, azimuth; zenith, nadir; altitude, colatitude, declination, right ascension.

1-5. Discuss: hour-angle, time zones, the date-line, (analemma?) chronometer, Greenwich mean time.

1-6. Discuss and construct a sun-dial. (horizontal base, vertical base, any base)

1-7. The earth is less a sphere than it is an oblate spheroid. (Explain oblate, prolate.) How was that determined?

1-8. Are vertical lines sometimes, always, or never parallel? (horizontal?; planes?)

1-9. (Chestnut) If the equator (idealized) is circled by a steel band which is then cut, opened, and an extra 6 foot length added -- how far will it be raised off the surface?

1-10. What direction is Moscow from here?
[INTO the earth!]

1-11. (Chestnut) I leave my tent, walk 10 miles south, shoot a bear, walk 10 miles east, then 10 miles north to the tent. What color was the bear? (Many solutions (?))

1-12. Explain: A straight tube without ends can contain any given quantity of water. A "bucket" to contain any given quantity of water can be made from a plane without any cutting or bending.

1-13. If you awakened in an enclosed room after an undetermined period of unconsciousness, could you tell if you were in the northern or in the southern hemisphere?

1-14. How can you determine direction on the surface of the earth? What is meant by "shooting the sun"? What observations are needed to determine position? Bring in some history of navigation and of search for accurate chronometer. (Why so important?) What is meant by "NORTH"? (Magnetic, Geographic, ...) Suppose that you stand at the North Pole and drop a snowball (along the axis of the earth). Does the ball travel north or south?

1-15. Some celestial "geometry". Plane of the ecliptic, signs of the zodiac, orbital plane, syzygy (transit, occultation, eclipse), parallax, precession, abhelion, perihelion, a parsec.

1-16. Many problems in cartography, large and small.

1-16.1 Various ways of representing (mapping) spherical surface on a plane: mercator, Lambert conical, polar, etc.; advantages, disadvantages.

1-16.2 Ordinary road maps can be used in a classroom in a number of ways: cordinates; scale representation (distance, area); direction; (vertical) surface features.

1-16.3 Contour maps, level lines. Determine vertical profile of a line across a contour map. Determine road route with minimum changes of elevation between given points.

2. The situations involving balances, levers, first moments, and so on lead to simple mathematical models, some involving easy linear equations and some involving more or less difficult puzzle situations.

2-1. On a single line the weights w_1, w_2, ... at distances d_1, d_2, ... are balanced by a given single weight W at what distance (D)? and so on.

2-2. A vertical cartesian plane with a set of weights w_1, w_2, ... at points P_1, P_2, ... is placed in neutral equilibrium by what weight (W) at a given point P ?

2-3. (As in 2-1) Find the center of gravity of a given weight-position distribution (on the line; in the plane(?)).

2-4. (Puzzles, old and new.)

2-4.1 A set of 9 coins contains one (light) counterfeit coin of identical appearance with the others. Determine, in two balancings, which is the light one. (Good for flow-chart analysis.)

2-4.2 A set of 13 coins contains one counterfeit coin of identical appearance with the others. (? lighter, heavier?) Determine in three balancings which is the counterfeit. (Excellent for flow-charts,)

2-4.3 Twenty sacks each contain 25 gold coins supposed to weigh exactly one pound each. One sack contains only counterfeit coins, each one ounce underweight. What is the least number of weighings to determine which is the sad sack?

2-4.4 Slippery Sam, the storekeeper, had various devices to cheat his customers: How?

2-4.41 IRON POTATO. A small misshapen "potato" made of iron.

2-4.42 IMBALANCE. A "balance" scale set slightly off-center.

2-4.43 LIGHT-WEIGHTS (heavy-weights). A set of brass weights plainly stamped "1 lb., 2 lbs., 5 lbs., 10 lbs.", but actually underweight (overweight) by 1 oz., 2 oz., 5 oz., 10 oz.

2-4.44 SHORT STICK (long stick). A ruler plainly marked "1 yard, 36 inches", but actually only 35 inches long.

* Dapper Dan is measured for a custom-made suit with a tape-measure marked "1 yard - 36 inches" but actually shrunk down to 35 inches. Will his suit be too small or too large? (That depends -- on what?)

3. Pendulums can lead to a number of interesting problem situations and mathematical models involving various levels of difficulty.

3-1. Foucault pendulum (rotation of the earth).

3-2. Simple harmonic motion (small oscillation).

3-3. Length-time relationship. Determination of g . Simple clocks; horology and chronometry.

3-4. A pair of pendulums (not in parallel planes) to show beats, etc.

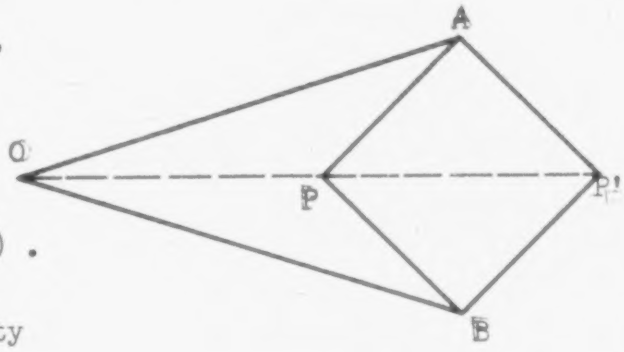
3-5. Lissajous figures, compositions of harmonic motions.

3-6. Combinations of pendulums: different lengths on same horizontal axis; different positions on the same vertical axis.

3-7. Isochrone, Brachistochrone, Tautochrone, (cycloid).

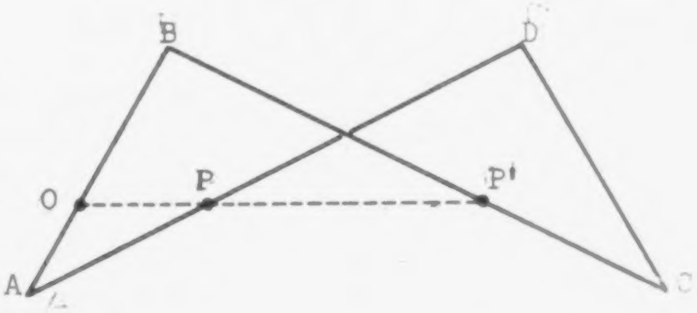
. Linkages.

4-1. Peaucellier: Rhombus APBP' , and equal links OA , OB . Show (1) O , P , P' are collinear; (2) OP · OP' = a constant $(r^2 = OA^2 - AP^2)$.



Because of this second property the points P and P' are inverse points with respect to the circle with center O and radius r , and this linkage is sometimes called Peaucellier's inversor. If O is fixed, and P traces any curve S , then P' will trace the inverse curve S' . In particular, if P traces an arc of a circle which passes through O , then P' will trace a segment of a straight line.

4-2. Hart: Crossed quadrilateral ABCD , with AB = CD and AD = BC ; also AP = CP' . Show: (1) there is a fixed point O on $\overline{AB}$ such that O , P , P' are collinear; (2) OP · OP' = a constant $(r^2 = (OB + BP') \cdot (AP - AO))$. (See comment: 4-1)



4-3. These linkages can be varied and combined:

4-3.1 In Peaucellier linkage connect the equal arms from O to symmetric points on $\overline{AP}$, $\overline{BP}$ and investigate OP · OP' .

4-3.2 A second Peaucellier linkage O'A'B' R,R' , with O',R on P,P' ; (other superpositions).

5. Topological problems.

These involve only incidence, separation, betweenness, connectedness, separation, and so on, but not any measure of length or area. They usually require more ingenuity than formal mathematics.

5-1. (Puzzle) Three houses are to be individually connected by non-intersecting lines to three wells.

5-2. (Puzzle) Unicursal curves; Konigsberg bridges; Hamilton paths.

5-3. Knots: classification, invariants.

5-4. Given an incidence matrix, construct (the) graph (equivalents):

5-4.1 In the rectangular area bounded by North and South Drives and East and West Roads there are five lanes that start at South Drive: Red, Yellow, Blue, Black and White; and five alleys that start at West Road: Gold, Silver, Lead, Tin and Copper. A lane intersects only alleys, and vice-versa. (Show that there can be no triple intersections.)

5-4.11 Draw a map if each lane intersects just one alley. (Possible?)

5-4.12 . . . . each lane intersects just two alleys. (Possible?) (just three? four? all five?)

5-4.13 If every lane intersected every alley there would be 25 intersections. If all lanes and alleys went Northeast there would be 0 intersections. Draw a map in which there are exactly $\underline{n}$ $(0 < n < 25)$ intersections. (Possible for all these $\underline{n}$?)

5-4.14 Given any specific incidence pattern, draw a map (Possible?), e.g., Red, blue; gold; yellow, black; silver; blue, white: lead; black, red: tin; white, yellow: copper.

5-4.15 Vary ad libidum: more lanes, alleys; non-rectangular boundary.

6. Originals (just for fun).

6-1. If P is inside square ABCD so that $m\,\angle PCD = m\,\angle PDC = 15$, prove that $\triangle ABP$ is equilateral.

6-2. Construct a cyclic quadrilateral, given the four sides. (Possible?) If possible, find 3 solutions with same lengths in various order.

6-3. Of all quadrilaterals with four given sides the cyclic quadrilateral has the greatest area.

6-4. If equilateral $\triangle ABC$ is inscribed in a circle, and P is any point on $\widehat{AB}$, then $PA + PC = PB$.

6-5. If regular hexagon ABCDE is inscribed in a circle and P is any point on $\widehat{AB}$, then $PA + PC + PE = PB + PD$.

6-6. What is the least cube that will just contain two unit spheres?

..... two spheres: one of radius 1, the other of radius 2 ?

..... three unit spheres?

(other combinations of spheres)

6-7. What is the least sphere that will enclose two cubes: a one inch and a two inch cube?

(..... other combinations.)

6-8. What is the height of the pyramid formed when a unit sphere rests on a base formed by three tangent spheres with respective radii 2 , 3 , 4 ? What is the least cube (sphere) that will enclose this pyramid?

APPENDIX B

Modeling - Grades 7-12

Contents

1. Teachers Manual for Supplementary Section 4M (Modeling) for Grade 7, titled "Applications of Mathematics and Mathematical Models".

2. Preface to Materials From the Modeling Group: Suggestions for Attacking the Problem of Making the Uses of Mathematics a Viable Part of the 7-12 Curriculum.

3. Suggestion for Carrying the Free Fall Example Begun in Grade 7, Chapter 3, on through Grades 8-9. This has been lifted from "Suggestions on Where to go with Flow Charting in Grades 8-9".

4. A Scaling Problem.

5. Hints for Teachers and Writers. Includes references.

6. Questions and Comments Raised by A Proposed Chapter on Physical Models for 7-8-9 Grade Level. This document is a concerted attempt to keep modeling in its proper place in the 7-12 sequence.

1. Teachers Manual for Supplementary Section 4M (Modeling) for Grade 7, titled "Applications of Mathematics and Mathematical Models".

This section has been printed separately from this report so that it can be tested in the classroom 1967/68.

2. Preface to Materials From the Modeling Group: Suggestions for Attacking the Problem of Making the Uses of Mathematics a Viable Part of the 7-12 Curriculum.

As everyone knows, it is very difficult to change school curricula and practices. Nowhere is this better illustrated than in the recommendations that the uses of mathematics be made to play a more prominent part in school mathematics instruction. Such recommendations have been a prominent feature of every major reform suggestion since 1900 (cf. E. H. Moore) with few visible results. From this we can either conclude that the problem is not solvable and simply throw up our hands, or we can hypothesize that we have simply not given sufficient attention to the problem in a sustained and consistent way.

It seems possible to subdivide the problem of how to make the uses of mathematics a genuine part of mathematics instruction into several distinct, and to some extent separable, parts, then see if each of these can be attacked. It should be apparent by now that any less systematic procedure will probably not succeed. As a start, let us try the following division of the problem, with each subdivision seen as further from solution than the previous one:

(1) Good problem material dealing with genuine "real" situations must exist. Furthermore, it must exist in sufficient variety to be usable for youngsters at any specified level in the curriculum. However, it is possible to believe that this is the least troublesome aspect of our problem.

(2) Once such material exists, it must be adapted and worked into the curriculum in sensible and fruitful ways. Existence is certainly no guarantee of effective use. For one thing, adaptation for specific

curriculum levels and purposes is nearly always necessary. For another, it is difficult to work in applications in natural ways that contribute to a spiral program, rather than appearing as isolated events.

(3) Even if good material on the uses of mathematics exists and is worked into the curriculum effectively there still remains the problem of whether, when, and in what way the processes involved in applying mathematics should be discussed explicitly. This problem has not yet engaged very many people, and among those who have tried to cope with it (e.g., the present outlining group) there are strongly divergent opinions. Even given such cogent descriptions of the processes as Burrington's and Juncosa's, it is not clear how much explicitness in these matters is called for in order to make the processes a natural part of students' awareness and functioning without becoming a memorized relatively useless catechism (such as "The Scientific Method or The Steps in Problem Solving").

(4) Given the solution to all the above problems in the material for students, teachers of mathematics may still find it difficult to use the materials in effective ways. The training of most mathematics teachers has not had much to do with either specific uses of mathematics or the processes through which mathematics is applied. Also, it may be that some of the new materials on applications will depend for effectiveness on modes of teaching that are quite unfamiliar to most teachers.

(5) How do we avoid such distortions of this emphasis as happened in the first round with respect to "sets"? That is, if we do aim for increased attention to the uses of mathematics and if we do try to make the processes involved explicit by, for example, adopting the rhetoric of "mathematical models" that is currently fashionable among actual users of mathematics, then there is certainly potential both for actual overemphasis and for journalistic distortion. There appears to be considerable (largely unexpressed) fear among the outlining group that this is likely to happen.

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If we consider these questions more or less independently, as I think we must, then perhaps we can sort out what we have in hand and what still needs doing. A preliminary attempt at this exercise is included below:

(1) *Existence of problem material*.

As far as mere existence goes, we are not nearly so impoverished as is sometimes supposed. The real problem is in the next category -- adaptation of existing suggestions and materials for our specific purposes. There is a need, of course, to collect more such applications material and groups should be convened to do precisely that, but not in a random way. At this stage, any groups convened to write problems or suggest applications should be given fairly specific assignments -- a problem is needed that motivates or applies a specific topic; a certain point about applications or building mathematical models needs to be made; an interesting context for certain finger exercises is needed; and so on.

Sources of raw material for the applications-mathematical models-will include at least the following:

(a) The New Orleans report has a long list of suggestions, many with specific references, on pages 3-14. It also has an annotated bibliography of several dozens of articles that deal with applications using only school mathematics and, in addition, the annotations specify what mathematics is used and the grade level difficulty of the articles.

(b) *SMSG Studies in Mathematics*, Volume XVI, contains reprints of 24 articles, including many of those annotated in the New Orleans report and some others.

(c) Many suggestions are contained in the July 1966 *Tentative Outlines of a Mathematics Curriculum for Grades 7, 8, and 9*. Other suggestions produced by the 1967 outlining group are contained in the present document.

(d) Such magazines as *Science*, *Scientific American*, and the *American Scholar*, as well as journals of mathematics organizations, operations research, *The Harvard Business Review*, and so on, frequently contain expository articles with material adaptable to our purposes -- or even usable directly. The annotations in the New Orleans report do not begin to exhaust the possibilities.

(e) The SMSG *Mathematics Through Science* and *Mathematics and Living Things* contain among them about twenty suggestions of simple experiments with rudimentary apparatus, the majority leading to graphing and curve fitting exercises.

(f) Circa 1953 and 1954 William L Schaaf published in the *Mathematics Teacher* a series of bibliographies covering various aspects of mathematics and its uses (e.g., "Map Projections and Cartography" M. T. (October 1953) 45:440-443). These contain references to a large amount of raw material -- the problem of selection and adaptation still remains. Similarly, P. S. Jones of the University of Michigan has compiled a number of such bibliographies and examples which he would no doubt share with anyone who wanted to take the trouble to go through his files.

(g) *Applications of Elementary Mathematics, A Compendium Prepared for The Mathematics Association*, London, G. Bell and Sons, Ltd., 1964, contains brief reference to a large number of applications. (In the SMSG library, No. C1256.)

(h) Textbooks of recent origin have many examples, especially in such fields as linear programming. See, for example, Dorn and Greenberg, and books on operations research, etc.

(i) Paperback books for popular consumption (e.g., some of the Sawyer books; William G. Vergara, *Mathematics in Everyday Things*, Signet T2098; and many others) have large numbers of simple minded applications perhaps suitable especially for the lower grades of our sequence. Again, the main problem is selection and adaptation to our specific use. It would be a tedious, but perhaps very useful, exercise to simply extract some of these on 5x8 cards, sort and classify them, and get them into the writing mill.

In other words, I think we already have more raw material on applications than we are likely to make effective use of. Any attempt to generate new material should probably be directed to quite specific topics and purposes.

(4) The Problem of Adapting the New Material on Applications and Working it into the Curriculum.

This seems to me to be obviously the crux of the matter and I think it is safe to say that no school text up to now has done it well. We might try to make visible at least the following specific types of materials:

(a) Applications that tie in with specific topics of our text. Most can be used either for motivation of a new topic or for application afterward to demonstrate its usefulness -- with the main line development independent of the application in either case. I prefer use as motivation, but one or the other should probably be a feature of most topics at present. Example: Argand diagrams of complex numbers in connection with electrical circuit phase relationships.

(b) Applications used to carry the main burden of development of a given topic. Example: linear programming as the reason for doing systems of equations (see appended material).

(c) Applications that use only elementary means to get at interesting or surprising results. Examples: Appended material on scaling laws via a new statue of J. T. Cornpone; Polya on the minimum popular vote to elect a president.

(d) Problem material arranged to make specific points about the process of applying mathematics. Example: Dual linear programming problems in appended material.

(e) Specific single topics carried over several grade levels in a spiral fashion. Example: Sequence on falling body laws as follows: [1] begin about as in Grade 7, Chapter 3, with Galileo's experiment; [2] continue as in Item No. 3 in this Appendix; [3] consider the behavior of material falling not in a vacuum (perhaps with parachute jumper lead in) via "The Falling Sphere" experiment in SMSG Mathematics Through Science, Part III, pp. 59-.

(f) A fairly global topic worked in over the whole 7-12 sequence. Example: Greenhood, David, Mapping, University of Chicago Press, 1964 (Paper). This is a beautiful book with material that could be worked in over the whole 7-12 sequence, from coordinates to contours and level curves to a rich variety of projections. The mileage one could get out of such material in considering applications via building models seems to me enormous, as well as working in nicely with many mathematical topics in our sequence.

(g) Applications kits where a problem is described to the youngsters but the pertinent data, means of solution, etc., is given only to the teacher. This would allow considerable variation in fitting instruction to the ability of the kids -- a teacher would simply be judicious in how much information to give out to keep the problem going. Collection of data and lengthy computation, as well as mathematics beyond the power of a given youngster, could be short-cutted by prefabricated data and results to be dispensed judiciously by the "consultant" (teacher). (One of the 1967-68 writers of eighth grade material has advocated this approach and intends to produce some such "kits" or "blocks" of material.)

(3) The Problem of How and When to be Explicit About the Processes Involved in Applying Mathematics.

This is a problem on which only prejudices exist at the moment. Some feel that a graded set of explicit treatments should begin fairly early with "simple" things such as arithmetic as a vehicle; others feel that the processes ought to be left very much implicit until a fairly grand example is at hand -- one that has at least two possible mathematical models, for example. Let us try both appropaches and see what happens. My own prejudice is that even simple arithmetic, geometry, and uses of "formulas" give sufficiently rich scope for discussion of some parts of the process to be worth exploiting. What is needed for this is a consistent graded sequence to try out.

It does seem to me that in our materials for students the word "modeling" should never be used as a verb, however useful it may seem to us in our discussions. Rather, one should always use a specific phrase: "We are looking for a mathematical model", "These things go into building a model", etc. To say, ever, that we are "modeling" is so unspecific as to practically guarantee distortion.

The last two problems listed at the beginning of this note are remote from present concerns, since we are very far from having solved the first three. The problem of teacher re-education for new approaches and a new emphasis is, of course, likely to be very sticky indeed. Perhaps we will have a better idea how to attack it as we make more progress on really good materials for students and think about the teachers' role in presenting such materials.

The appended materials do not represent a consensus, but are merely a series of individual contributions as a few people thought about the problems of getting uses of mathematics into school materials. Some of the appended materials are referred to above, others are not.

3. Suggestions for Carrying the Free Fall Example Begun in Grade 7, Chapter 3, on Through Grades 8-9.

This has been lifted from "Suggestions on Where to go with Flow Charting in Grades 8-9".

A Note on Modeling

Mathematics cannot deal directly with physical objects. Mathematics can only talk about idealized objects such as points, lines, numbers, and functions. These objects are abstract creations of the mind and have no existence in the real world. In order to use mathematics to solve problems about real life objects, we must first create a "Mathematical Model" in which the real life objects are represented as mathematical objects.

In different types of problems, the same physical objects may be represented mathematically in different ways. For example, when we draw geometric figures on a sheet of paper, we think of the sheet of paper as representing a plane. However, when we have a problem involving the volume of a book, we think of the sheet of paper as a box shaped solid with one dimension very small compared with the other two. We would say that we have chosen different mathematical models appropriate to the different types of problem.

Let's consider some problems concerning gravitational attraction. We have already considered the problem of falling objects and Galileo's experiment. Now we want to take a look at the modeling process involved.

In thinking of Galileo's experiment, we think of the falling objects as points. More accurately, we think of the locations of the objects as being points. We regard the surface of the earth as a plane and we think of the paths of the objects as being parallel lines, both perpendicular to the surface of the earth.

We will disregard air resistance. This amounts to assuming that the objects are falling in a vacuum. We will assume that the distance traveled by a falling body in a given time does not depend on the height from which the object is dropped.

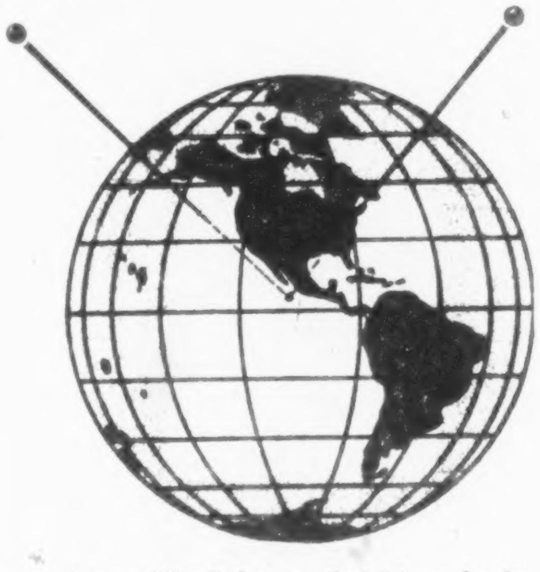
We finally assume that the distance travelled by a falling object is given by the formula:

$$d = 16\, t^2$$

where t is the time in seconds and d is the distance in feet.

A rather strange picture of the world! The earth is a plane with nothing but vacuum above it and a falling object is squeezed down into a single point.

In fact, every one of our assumptions is wrong. We know that the earth is roughly spherical in shape and that falling objects will fall toward the center of the earth and their paths will not be parallel.



Furthermore, the distance travelled by a falling body in one second is not independent of the height of the starting point. Even if we neglect the effect of air resistance, an object falling from a mile high will fall less far in a second than an object dropped near the earth's surface. The amount less would be about one part in 2000 .

Air resistance is certainly not always negligible. It is because of air resistance that a piece of paper falls more slowly than a penny. They would fall at the same speed in a vacuum. (See SMSG Mathematics Through Science, Part III, pp. 50+, for an experiment showing that velocity becomes constant for fall in a resisting medium.)

All of these remarks must have weakened your confidence in our model. That was what they were supposed to do. Now we are going to restore your confidence again.

Although the earth is a sphere, it is such a big sphere that a small portion of its surface is very nearly a plane. If two objects fall to the earth so that they land no more than 100 feet apart, then their paths miss being parallel by about $\frac{1}{3500}$ of one degree, which is practically negligible.

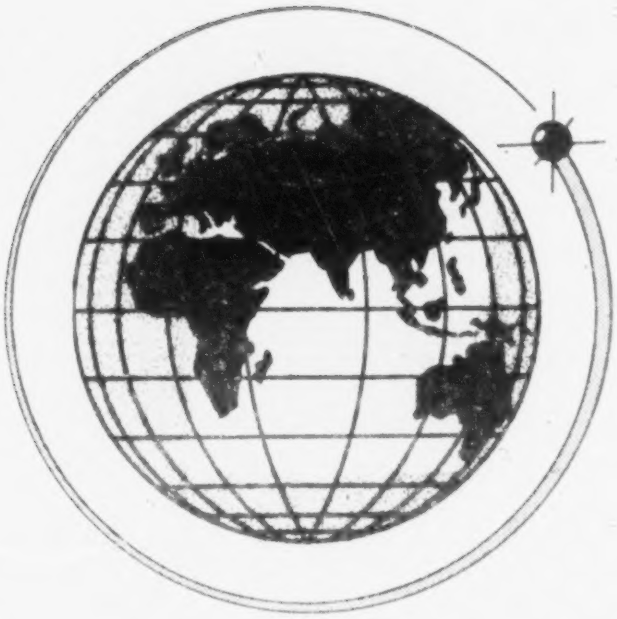
The effect of the height of the starting point only produced a difference of one part in 2000 for objects dropped from a mile high. The effect will be even more negligible if we consider only objects dropped from within a few hundred feet of the earth's surface.

The effect of air resistance is very complicated. It depends on the weight, shape, and the speed of the falling body. For objects which are nearly spherical in shape and about as dense as a rock falling for no more than two or three seconds (so as not to build up too much speed) we can regard air resistance as negligible.

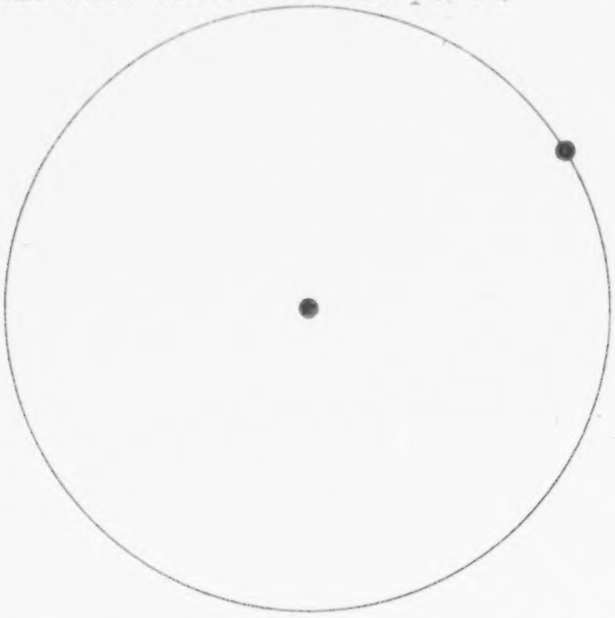
So our model for the motion of falling bodies is not so bad after all. In fact this model is used for very accurate scientific calculations involving "in the small" or "local" problems. In such work, however, the more precise formula $d = 16.1\ t^2$ is used instead of $d = 16\ t^2$.

Another type of gravitational problem concerns a satellite travelling in an orbit around the earth. Just as the falling body falls due to the earth's gravitational attraction, so the satellite is held in its orbit by the earth's gravitational attraction.

It would be ridiculous in such problems to represent the earth as a plane. Instead, we would represent the earth as a sphere and the satellite as a point moving around the sphere in a circular path.



It is interesting that a simpler model gives the same results in many cases. In this simpler model, the earth is represented by a point and the path of the satellite by a circle with center at this point.



This amounts to considering all the mass of the earth to be compressed into a single point at its center.

To get more accurate results, more "sophisticated" models are required (i.e., results agreeing more closely with observed behavior) Near the end of the 17th century, Isaac Newton devised a wonderful model for describing the motion of objects. The model was so good that for 200 years no phenomena were observed which did not agree with this model. Scientists began to forget that they were dealing with a model. They thought that they were actually talking about the real world. They felt that all motion had to be governed by

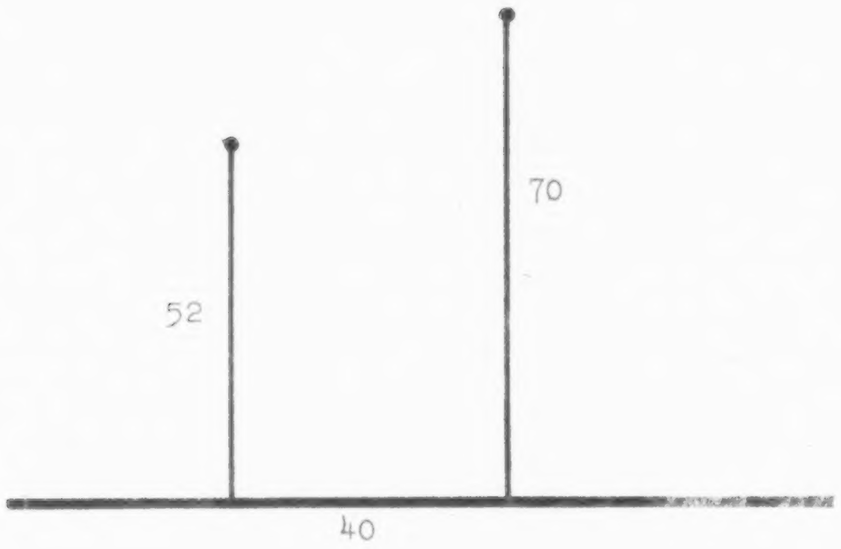
"Newtonian mechanics", i.e., by the functions used in Newton's model. But around the end of the 19th century a number of experiments seemed to indicate that some phenomena did not agree with the Newton model. For a period, many scientists made vain attempts to explain these phenomena in terms of Newtonian mechanics. Finally, Albert Einstein devised a new model which explained the mysterious phenomena. Calculations involving this model led to some predicted behavior which seemed wildly improbable. For the first forty years of the present century experiments were devised to determine whether the predicted behavior actually occurred in fact. These experiments confirmed the predictions of the Einstein model. Among the consequences was nuclear energy. Today, all sophisticated work involving high speed particles is done in terms of the Einstein model. In some calculations, however, the added sophistication of the Einstein model is not needed and the old Newtonian model is used. The physicist must know which model is appropriate to his problem.

Our process in finding a mathematical solution to a physical or real life problem can now be described. First, we select a model appropriate to the problem. Then we make exact calculations relative to this model. All of our answers are relative to our model. We do not concern ourselves with the question of how closely our results agree with the real life situation. That is not to say that the answers to such questions are of no interest to us. But the answers to these questions lie in the realm of physics, not mathematics.

To see how this modeling process works, consider this problem.

> Problem: Two towers are 40 feet apart. One is 70 feet high and the other is 52 feet high. A stone is dropped off the higher tower and one second later a stone is dropped from the shorter tower. How far apart will the stones be one second after the second stone is dropped?

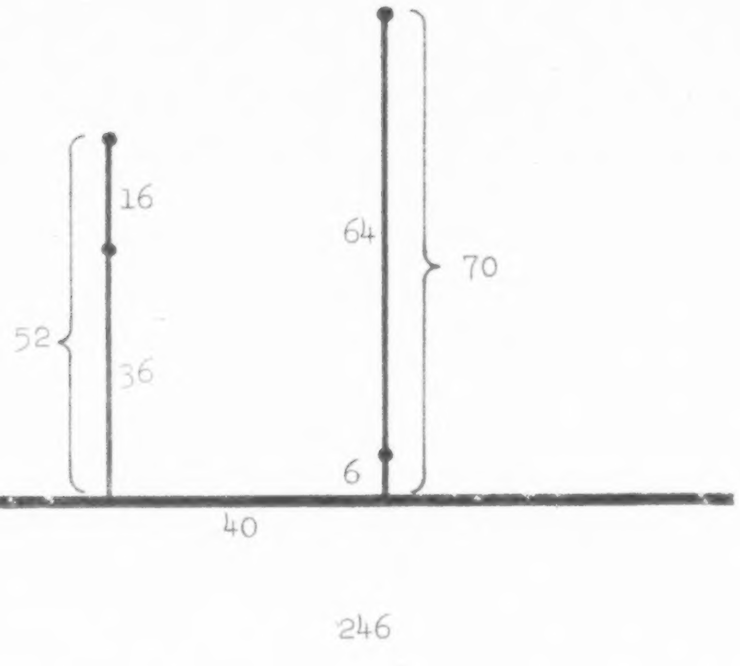
Now we will discuss the solution to this problem. No mention has been made of a model but it is tacitly understood that the falling body model discussed earlier in this section is to be used. We draw this sketch.



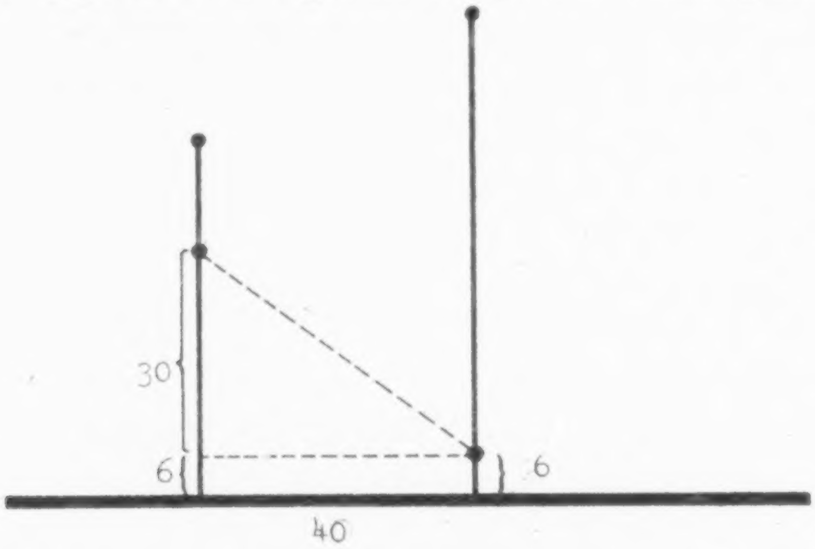
The horizontal line lies in the surface of the earth. The vertical lines represent the paths of the two stones. If we knew where the two stones were at the desired time, we might be able to find the distance between them. We use the function

$$S : t \rightarrow 16t^2$$

to get the distance fallen by each stone. The first stone dropped for two seconds and travelled 64 feet; the second stone fell for one second and travelled 16 feet.



We see that the first stone is 6 feet above the ground and the second stone is 36 feet above the ground at the time in question. If we draw a horizontal line at height 6 feet above the ground we see that the required distance is the hypotenuse of a right triangle whose legs have length 30 feet and 40 feet. This distance is easily computed from the Pythagorean theorem to be



$$\begin{aligned} d &= \sqrt{30^2 + 40^2} \\ &= \sqrt{900 + 1600} = \sqrt{2500} = 50 \ . \end{aligned}$$

The correct answer to the problem is 50 feet. It is the only correct solution in our model. If you went out and performed the experiment, and measured the distance, and came out with 52.3 feet, that answer would not be considered to be the correct answer. You are supposed to be working in the falling body model. We are not supposed to accept the answers that come from the model, regardless of how closely they agree with reality.

4. A Scaling Problem.

Problem for class discussion on similarity and its implications involving scaling laws, modeling (in an explicit sense) and approximating assumptions. (Language is expected to be adjusted to level expected of students.)

Consider the following problem:

The municipality of Dogpatch (begging Al Capp's permission) wishes to erect a new and perhaps different cast bronze equestrian statue of its famous

Civil War hero, Gen'l. Jubilation T. Cornpone, the previous one having disappeared into the last batch of Kickapoo Joy Juice made by that inveterate moonshining pair, Lonesome Polecat and Hairless Joe. The "statchoo" committee, headed by none other than, you guessed it, Mammy Yokum, is a little concerned about the cost and would like to at least try to estimate the cost of the material. Now, it just so happened that one citizen, a nephew of Sen. Jack S. Phogbound (who else?), did actually get as far as completing the seventh grade in school and he was appointed a subcommittee of one to estimate the cost. How may he go about doing it?

During his data gathering stage he gleaned the following information:

- The statue was to be twelve feet tall from the tip of the General's upraised sword to the bottom of the hooves of his steed.
- The unit price of bronze is $1.00 per pound.
- Bronze statues are hollow shells with nearly uniform thickness and for bronze statues of approximately the intended size and shape the average thickness of the shell is one inch.
- The artist's clay model of the statue was two feet high when measured in the same way.

Since the information gathered about the statue is essentially dimensional, whereas the cost information is in the form of price of bronze by the pound, he realized he needed some information connecting weight and volume for bronze. Thus he also sought and found that

- A cubic foot of brass weighs 400 lbs. (The weight per unit of volume of a substance is called its density.)

Now, he knew that if he could only determine the volume of the intended monument in cubic feet simply multiplying the unit price by the density and this product by the volume would give him the total material cost of the statue.

Lowering the scale model in an irregularly shaped vat he was able to commandeer, he filled the vat to the brim and then drew out the model, taking care not to lose much water in the process. By carefully measuring the water added to fill the vat again to the brim he was thus able to get the volume of the artist's clay model of the desired monument to "Jube". It was 4 cubic feet. But, how did he get from this volume of a solid figure to the volume of a relatively thin, approximately uniformly thick shell whose external shape is similar? Noting that, since the shell is thin and essentially uniform in its thickness, the interior volume contained by the shell must also be essentially similar to the clay model, he reasoned that the volume of the shell could be obtained simply as the difference between this interior volume and the total volume enclosed by the outside surface of the statue.

Now, how did he obtain these volumes from his previously obtained information? By use of certain "scaling laws" for similar volumes. Let us see what these "scaling laws" are.

Recall various surface area and volume formulas from past, e.g.,

$4\pi r^2$ = surface area of sphere of radius r.

$\frac{4}{3}\pi r^3$ = volume of sphere of radius r.

By calculation, observe that doubling radius, quadruples area and multiplies volume by 2^3; tripling radius multiplies area by 3^2, volume by 3^3.

Lead to fact that multiplying radius by p, multiplies area by p^2 and volume by p^3.

Consider cylinder $A = 2\pi rh + 2\pi r^2$, $V = \pi r^2 h$.

Double radius and height (i.e., preserve similarity) and note again quadrupling of area and "octupling" volume and multiplying radius and height by same factor p leads to area multiplied by p^2 and volume by p^3.

Scaling Laws

Consider a few more examples, e.g., cones and lead heuristically to conclusion that if any three dimensional figure is scaled by a factor p (i.e., any linear measurements taken on it are multiplied by a factor p) then its area is multiplied by a factor p^2 and its volume by p^3 or stated as ratios

$$\frac{A_1}{A_2} = \frac{\ell_1^2}{\ell_2^2} \quad \text{and} \quad \frac{V_1}{V_2} = \frac{\ell_1^3}{\ell_2^3}$$

where ℓ_1 and ℓ_2 are any two corresponding lengths in the two figures and A_1, A_2, V_1 and V_2 are the respective areas and volumes.

(Question! Has the student been exposed to indexed variables before this time?)

Returning to the problem given the Honorable Senator's nephew's problem, the chronicles have it that, in possession of these facts he computed the volume V_o to be encased by the outer surface of the intended monument from

$$\frac{V_o}{4} = \frac{12^3}{2^3}$$

giving 864 cubic feet. Similarly, but with more arithmetic labor the interior volume V_i encased by the shell, from

$$\frac{V_i}{4} = \frac{(11\frac{10}{12})^3}{2^3} = \frac{(12 - \frac{1}{6})^3}{8} = \frac{1}{8}\left(1728 - 3 \times 144 \times \frac{1}{6} + 3 \times 12 \times \frac{1}{36} - \frac{1}{216}\right)$$

he obtained $V_i \doteq 828.5$ cubic feet ($\doteq$ means approximately equal) and finally the approximate volume of the statue as

$$864 - 828.5 \doteq 35.5 \text{ [illegible]}$$

Thus, the approximate cost of the bronze was determined to be $14,200.
(I hope the arithmetic is correct.)

* * * * * * * * *

Simple follow-on problems for the student:

1. Suppose that the surface area of the model of General Gompers's statue was 15 square feet and as an afterthought it was wondered what it would cost to gild the statue with $\frac{1}{200}$ inch thick gold foil which costs $50 per ounce and whose density is three times that of bronze. What would the cost of materials be?

2. Approximately how much longer would it take for one man to paint the exterior of two essentially similar buildings with essentially equal ease of accessibility of its surfaces when the heights are in the ratio 3:2.

3. Which is worse: to be slugged during a riot by a crude blackjack made of a sock containing 4 steel ball bearings each 1/2 inch in diameter

or by one containing one bearing with a diameter of 2 inches?

4. Joe lives 1 mile from the main street where the trolley cars run every 10 minutes. Their average speed, because of stops, traffic, passengers fumbling for change, etc., is 10 miles per hour. To go to school he must travel 5 miles on the trolley car, besides his walk which he can do in 15 minutes.

For the summer he has taken a job at the beach. This requires him to take the railroad from a station which is 2 miles from his house and 8 miles from work. The train's speed, however, averages 20 miles per hour. Joe decides to jog the 2 miles from his house to the station in fifteen minutes to save time.

What should the time between trains be for his average total travel time to the beach. Is it the same as to school? What assumption did you make about Joe's departure, to arrive at this answer? What other assumptions could you make? What effect would they have on your answer?

5. A meteor coming into the earth's atmosphere takes on a quantity of heat per unit time which is proportional to its surface area. This quantity of heat per unit time, in turn, is proportional to the volume of the meteor and to its temperature rise per unit time.

Suppose another meteor of similar shape and same material but of twice the volume of the first were to crash into the atmosphere also at the same speed. What effect would its size have on its temperature rise per unit time? What is the ratio of the two temperature rises?

7. Hints for Teachers and Writers.

If it is our task to teach students mathematics, if it is our goal to prepare students to handle situations which will confront him later and if it is our desire to show him the way toward such a goal in an effective yet enjoyable and exciting manner, as he moves along from Grade 7, we must plan his activities carefully. There must be student involvement, class discussion, an atmosphere in which questions may be raised by students or teacher and answers developed as a result of this give and take.

To this end, throughout the course modeling (as treated in the "Outlines July 1966", see pages 16-30), should be used in a natural way. The "teaching-problem-solution" sequence must be replaced by creating a problem situation where the student takes over or at least participates in analyzing the situation, formulating the problem and finding a solution. He must recognize the relation between the problem and mathematics. Starting with simple situations and without making him aware of modeling at first, this approach should be ** developed and become an aid in his thinking. Then, when his mathematical inventory allows, there must be made a conscious effort, in a systematic way, to show him the important role of models in various situations: mathematical, physical, life.

(1) Examples:

(a) At a unit price of m dollars, the cost c of n articles is

$$c = mn .$$

(b) At the rate of m miles per hour, the distance d travelled in n hours is

$$d = mn .$$

(c) If the number of students per class is m, the number B of textbooks needed in n classrooms is

$$B = mn .$$

All three of these problems are models of linear equations.

This phase of teaching needs no special emphasis of the fact that there is a model involved.

(2) When in Chapter 3 the function concept is introduced and a problem leads to mapping $x \to ax + b$, as before, we graph $y = ax + b$. The graph is a description, a model, of the linear equation, which allows for a meaningful discussion of input-output, effect of the factor a on graph, etc. Again this is still the phase where little emphasis on modeling per se is made. But we, as teachers, must be aware of it and a casual

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** This, of course, holds only if section [illegible] was omitted. The teacher should definitely study this chapter.

remark might be made here and there.

(3) Now consider the problems in Grade 7, Chapter 8, Congruence, pages 1 and 24. Lead the student to recognize (and the solution should not be in the text) that he can sight both towns from a point. How do we go about the solution? You need a geometric model, in this case a triangle. By the proper questions the student should be led to this stage. After drawing the triangle the next step is to find a method for the solution.

The ways in which the students locate the point from which the towns "can" be sighted should lead to different triangles. Solutions by actual measurements, using congruent triangles, similar triangles and scale drawings should be encouraged; different results should be compared and interpreted.

Use the same approach for Example 1, Section 8-5.

-- again the text should contain no more than the first 5 lines and the diagram (OMIT the solution from student text).

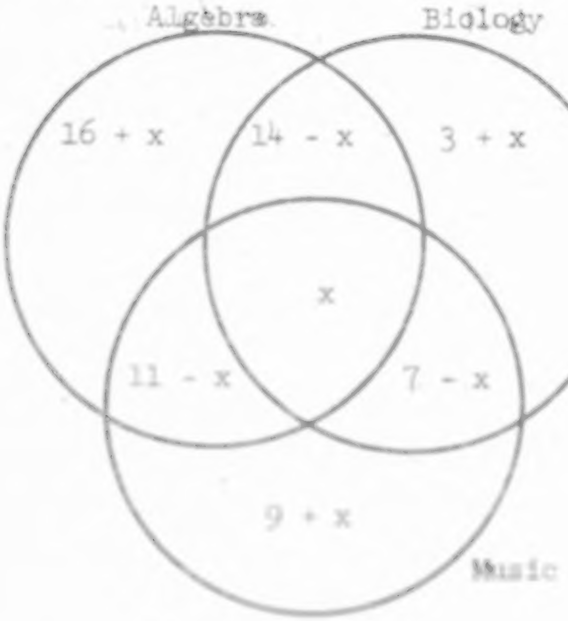
(4) Next let the students create some situation for which a similar model can be constructed. Divide the students in groups A, B, C, etc. and let group A find a solution for a situation developed by group B, etc. Possibly the students will think up a problem for which they can devise a model but then they are not ready, mathematically, to go on -- they are not prepared to solve the equations, constructions, etc., involved. Do not reject such a situation but capitalize on it, motivating further study in mathematics.

Now it is time, in a next round of modeling, to expose the students to a problem situation which is more involved (read pp. 13-19 of [illegible], July 1966) but within their reach.

Example 1. (Grade 7, Chapter 7)

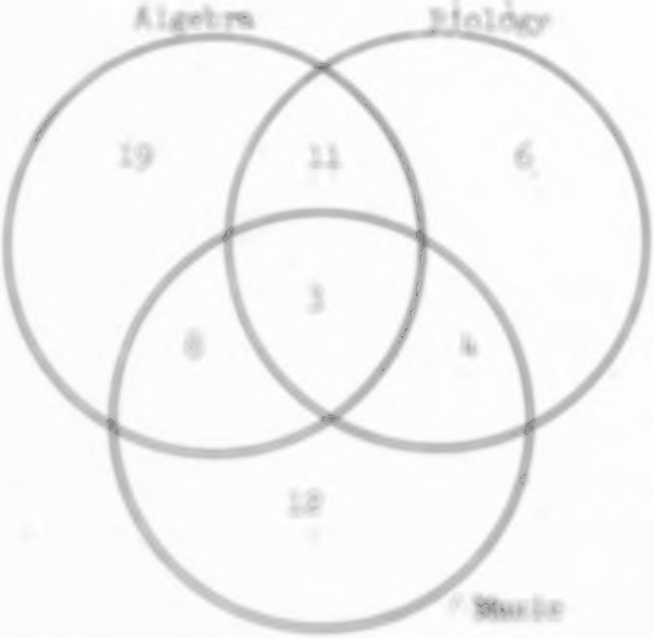
There are 65 students of whom 41 take algebra, 24 biology and 27 music. If 16 students take both algebra and biology, 11 algebra and music and 7 biology and music, how many are enrolled in one, two, or all 3 subjects?

Suggested Solution. (For teacher only): **Let the number of students taking all three subjects be x.


Algebra
Biology
16 + x
14 - x
3 + x
x
11 - x
7 - x
9 + x
Music

$$41 + 19 + x = 63$$
$$60 + x = 63$$
$$x = 3 .$$

There are 3 students in all three subjects. Now go back to the Venn diagram:


Algebra
Biology
19
11
6
3
8
4
12
Music

** See problems 2 and 3 in Chapter IV, Part II.

From the point of view of modeling let us trace the various transitions from model to model, concrete as well as conceptual.

(a) concrete (students)
↓
(b) conceptual (numbers)
↓
(c) concrete (Venn diagram)
↓
(d) conceptual (equation and solution)
↓
(e) concrete (interpretation of result as numbers of students).

Compare the aid to thinking by step (c), using the Venn diagram with the solution, possibly suggested by students, going from step (a) to step (b) to step (d): Let the number of students studying algebra alone be a, studying biology alone be b, studying music alone be c, studying all three subjects be x.

Then:

$$\begin{cases} a + b + c + 11 + 14 + 7 - 2x = 63 \\ a + 11 + 14 - x = 41 \\ b + 14 + 7 - x = 24 \\ c + 11 + 7 - x = 27 \end{cases}$$

This system is obviously beyond the student at this stage -- 4 equations in 4 variables.

On the one hand emphasize the importance of creativeness, the power of a good idea, the clarity and brevity of the solution offered by using a Venn diagram. On the other hand the system of 4 equations can be solved by the children at this stage -- with a little imagination. They learned the properties

if $p = q$ and $r = s$ and $x = y$,
then $p + r + x = q + s + y$,
and $m + n - p = (m + n) - p$.

Applied to the system of 4 equations (by adding the last three equations) you get the system

$$\begin{cases} (a + b + c) - 2x = 31 \\ (a + b + c) - 3x = 28. \end{cases}$$

Now let the number $(a + b + c)$ be n ,

then
$$\begin{cases} n - 2x = 31 \\ n - 3x = 28 \ , \end{cases}$$

which in turn is equivalent to

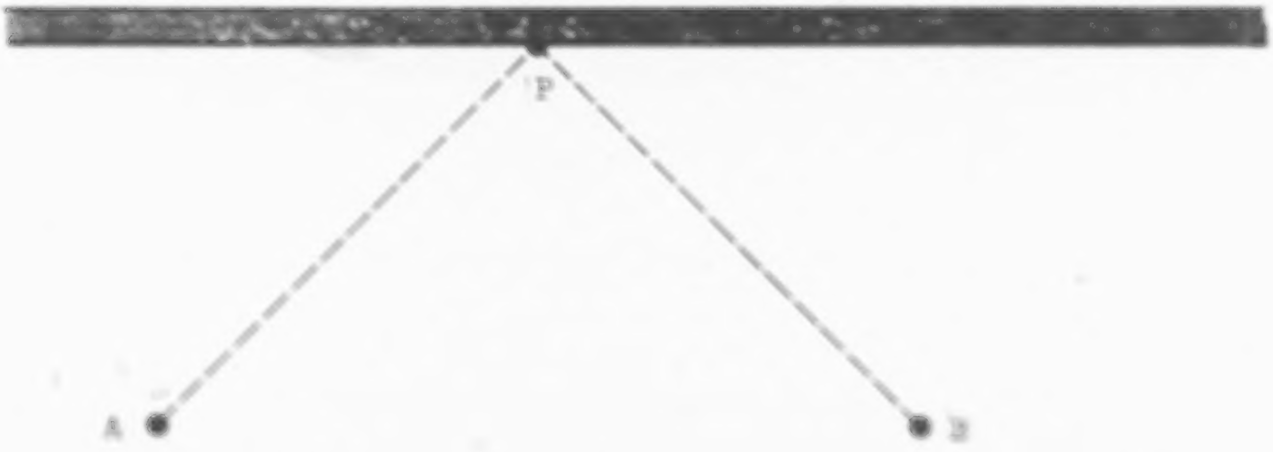
$$\begin{cases} n = 2x + 31 \\ n = 3x + 28 \end{cases}$$

so
$$3x + 28 = 2x + 31$$
$$x = 3 \ .$$

It is important, whenever possible, to show that one might find a way to handle a "difficult situation" but also, since this system is a very special one (the coefficients are very cooperative) this system can well be used as a motivation for solution of systems of equations.

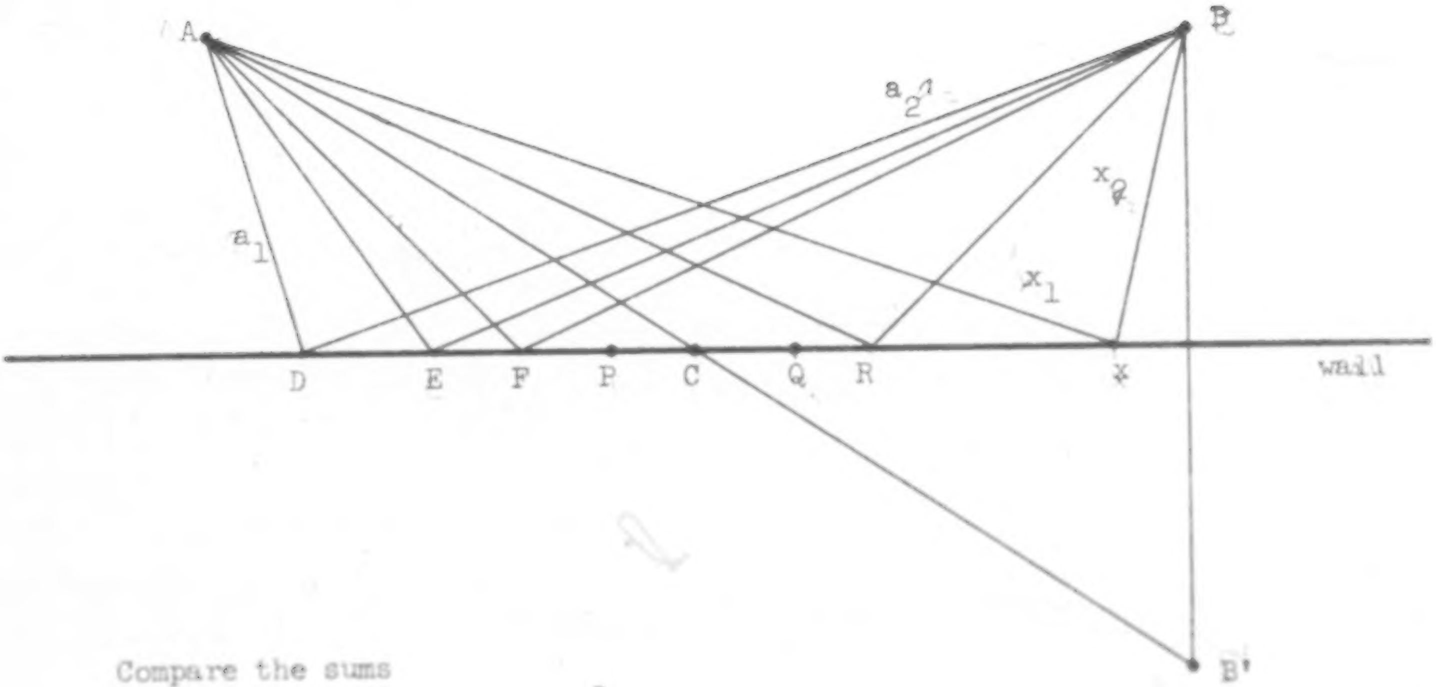
Example 2. (Reflection example, Grade 7, Chapter 8)

Race: Run from A to B , touching the wall, in as short a time as possible.



Solution. (For teacher *only*, not for student text)

(a) Compute the sum of the distances from A to P to B for various cases. The shortest distance will require the least time. Let the students measure the distances and tabulate for various cases. (Call the numbers s_1 , s_2 , ...)



Compare the sums

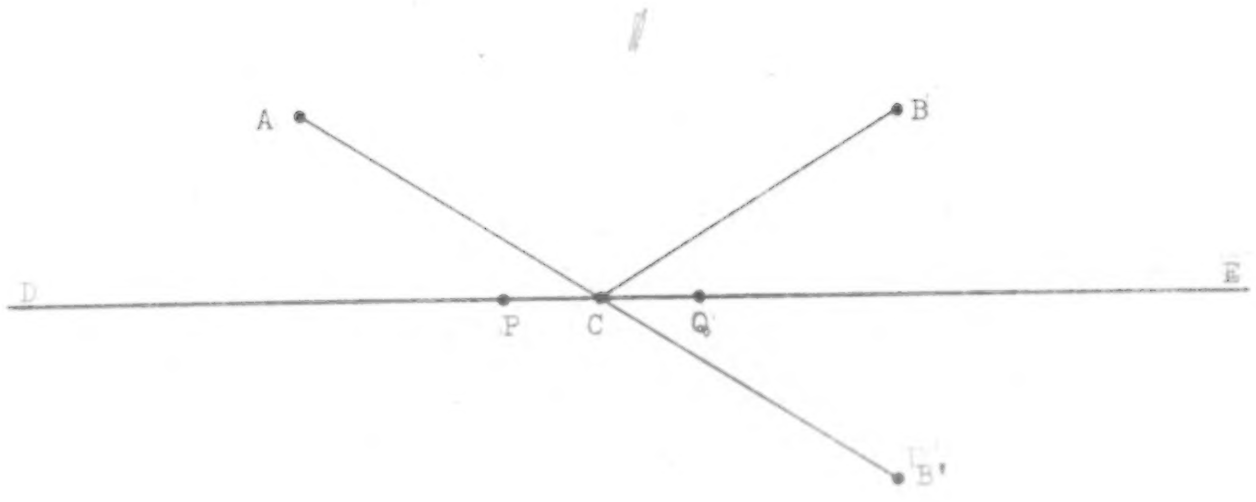
$$\left.\begin{array}{c} a_1 + a_2 \\ b_1 + b_2 \\ c_1 + c_2 \\ \vdots \end{array}\right\} \text{decreasing}$$

$$\left.\begin{array}{c} r_1 + r_2 \\ s_1 + s_2 \\ \vdots \\ x_1 + x_2 \end{array}\right\} \text{increasing}$$

Observe the sums get smaller, then again larger. Is there a point on the wall for which the distance is less than for any other point?

(b) Conclude conjecture:

There exists an optimal point C (it must be between P and Q).



(c) Conclude also conjecture that

$\angle ACD = \angle BCE$ at optimal choice of C .

(d) Show how use of mathematics can prove the general theorem by using reflection, vertical angles, shortest distance between two points, (A , B') $\overleftrightarrow{AB'}$ is a straight line and B' the reflection of B in line $\overleftrightarrow{DE}$.

(e) Note how mathematics saves computation although the model suggests the theorem.

Next:

See Appendix A -- Problem Situations Leading to Geometric Models, pp.

Suggested Grade and Chapter Level:

1. Grade 7, Chapter 8 , (a) - (e); (b) more difficult!

2. Grade 7, Measurement Chapter, (a) and (b); Grade 8, Linear Prog., (c) - (e).

3. Grade 7, Chapter 8, (a) - (d); (b) Any level of Geometry.

Also:

See Appendix A -- Problem Situations (which lead to geometric models), pp.

References:

1. Richmond, Prof. Donald E.: Mathematical Models of Growth and Decay, Twenty-Eighth Yearbook, National Council of Teachers of Mathematics, 1963.

2. Magee, John F.: Guides to Inventory Policy, I Functions and Lot Size (particularly, Optimum Lot Size, pp. 56-60), Harvard Business Review, Vol. 34, No. 1, Jan.-Feb. 1956, pp. 49-60.

6. Questions and Comments Raised by A Proposed Chapter on Physical Models for 7-8-9 Grade Level.

It is assumed that such a chapter would come after a reasonable amount of mathematical vocabulary, discussion of numbers, points, graphing, functions, mensuration, elements of geometry, and exhibition of some mathematical symbol manipulation has been put to the students. Nevertheless some problems remain.

Manipulative skills and sophistication could still be at a very elementary level; after all, the student has come out of arithmetic only the year before. Thus the examples must be simple. However, if they are too simple the spirit of modeling is not grasped. It is essential that neither the teacher nor the student become partial to the idea that modeling is just a new name for procedures for specific problem solving. It is important that they realize that it is a tool, or even more, a methodology, for facilitating thinking about a problem or classes of problems, which problems or classes thereof need not have a basis in the physical or non-mathematical world. (In fact, the use of an artificial physical world model is most common in clarifying probabilistic and combinational questions particularly through examples involving drawings from urns or, in other cases, random walks. Similarly, the use of Venn diagrams in problems having some set-theoretic content is another case of a physical model for a mathematically structured problem.)

A feasible area ripe with opportunities for model making is mathematical programming, particularly linear programming. Models in economics, industrial planning, network routing, operations analysis, etc., having mathematical programming formulations abound. However, if in the earlier chapter on linear algebra, specializing in linear equations and inequalities, also briefly introduces linear programming, then a re-visit may tend to give the teacher and student the idea that modeling is solely the solution of certain large scale problems by mathematical programming. Thus caveats that these examples are only one kind of modeling, or in this case, equivalently, problem formulation, what goes on should be frequently emphasized.

Another question raised concerns the level of notational sophistication achieved by the student before his exposure to linear programming. For problems involving several variables double subscripted coefficients are likely to occur and the student should have been previously exposed to them -- likewise for notations such as $P(x,y)$ and $P(x_1, x_2, \dots, x_n)$. This prevents foundering on the reefs of manipulative difficulties while attempting to arrive at the port of understanding.

Another area where mathematical models of physical situations abound is probability. (In fact it is the physical content of probability that makes for its existence as discipline, whereas, mathematically probability is axiomatically just a special case of measure theory with a fair amount of manipulative machinery from other branches of analysis.) Again the student needs some familiarity and sophistication in symbol manipulation if the examples of modeling here are not to be too simple. Problems which translate immediately into mathematical models and their concomitant formulas do not convey the spirit of modeling beyond a very shallow level.

Plane Euclidean geometry has provided the earliest example of a large scale mathematical modeling activity and one would expect that it would provide a continuing source of examples of modeling. Nevertheless, it does not seem comfortably feasible to develop an entire chapter at an early stage devoted to exposing the spirit, goals, and methodology of model making. It is possible, however, that certain items for class discussion (something less than a chapter) can be developed. An example is the statue problem for class discussion on similarity and its implications for scaling laws, etc. Here modeling, with explicit stating of simplifying assumptions, (1) enables the student to think about an unfamiliar situation, bringing the problem down to a level of hope that it is soluble; (2) identifies the facts that need to be known or determined; (3) suggests the requirement of scaling laws subsequently heuristically derived; and finally (4) actually enables an adequate solution to the problem.

Similarly at an early level going to simple physical kinematic problems for examples of modeling results in too quick a transition from the physical situation to formulas to give the right depth. It is essential that the teacher and the student do not look upon the activity in much the same light as they may the problems at the ends of chapters.

These remarks are intended to suggest that although understandable remarks and digressions on models and modeling and their various aspects should be liberally but not profusely nor oppressively (like salt [illegible]) sprinkled throughout the mathematical education process, a chapter on mathematical models of real world situations should be deferred until the 10-11-12 grade levels. This is to enable the student to get a reasonable intuition for model making by

(1) ensuring that the student has some reasonable mathematical vocabulary and familiarity with things such as subscripts, single and double, functional value notation, e.g., $f(x,y,z)$, composition of functions $f(g(x))$;

(2) allowing the students to have more extra-mathematical knowledge, e.g., in natural and physical sciences, economics, business, albeit of a very rudimentary sort to permit the formulation of problems which are reasonably plausible to the student;

(3) allowing the student to have command of some primitive mathematical skills with a little variety so that, if possible, two different mathematical modelings of the same problem leading to similar answers can be made, or even, if extrapolated far enough the two models could lead to substantially and possibly qualitatively different answers giving impressions of the strengths as well as limitations inherent in practically all mathematical modeling. (One sees this in the closeness of results obtained with plane and spherical trigonometry used on problems involving points relatively close on the earth's surface as contrasted with the absurdities obtainable from the use of plane trigonometry on widely separated earthpoints.) (A reasonable bag of prerequisites may include knowledge of how to solve systems of linear equations, sum arithmetic and geometric series, evaluate some polynomials, graph functions, compose functions of functions, solve quadratics, possibly some knowledge of elementary combinatorics and probability, no fear of summation symbols or of seeing, say, $\sin(x_1)$ for the first time.)

Furthermore, whenever or wherever a chapter or lesson, say a class discussion, on models or model making (thinking primarily of the case of a mathematical model of real world situation) the guidelines made in the Report of the Modeling Committee, pp. 17-30, of the *SMSG Tentative Outlines of a Mathematics Curriculum for Grades 7, 8, 9*, July 1966, are worthwhile adhering to.

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A chapter on mathematical models should have a variety of examples or, more appropriately, something resembling case histories. Possible contents can include one or two linear programming problems, a simple dynamic programming problem (good for iteration and flow charting as well), a queuing problem or two, some deterministic and probabilistic growth and decay situations. (Problems in biological, economic, and gambling or random walks exhibiting isomorphic structure are desirable. A strategic problem or two formulable as game theoretic problems. Possibly even a market situation may be conceived of here.) (Note different criteria, maximizing expectation, minimizing maximum possible loss, etc., frequently lead to different models and different conclusions.)

APPENDIX C

Systems of Linear Equations and Inequalities via Linear Programming

This Appendix is a Chapter on Solutions of Systems of Equations and Inequalities for some grade level 7, 8, or 9.

Contents

1. Introduction.

2. The Mathematical Model.

3. Related Problems.

4. Outline for Mathematics Sections of Chapter on Systems of Linear Equations and Inequalities.

5. Suggestions for the Development of the Topic, Systems of Linear Equations and Inequalities in Grades 7, 8, 9, and 10.

6. Some Thoughts on the "Student's Manual" to Accompany the Teacher's Text for Model-Motivated Mathematics (3M).

7. Two Pairs of Physically Stated Dual Linear Programming Problems.

Background Assumptions.

Same as p. 149, July 1966 Outline Book.

Content.

Introduction to systems of linear equations and inequalities with optimization.

Purpose.

To teach the mathematical content through immediate problem involvement and modeling.

The introductory section is designed through a decision problem to motivate and provide sufficient familiarity (using only arithmetic and logical reasoning) to permit the development of a mathematical model of the problem.

1. Introduction.

Often in life we must make a decision. Sometimes the decisions involve things that are not important or are easy to make, like what time to get up or what to have for dinner. Sometimes they are very important and very difficult to make, like what job to take or where to live. Think of the decisions you already made today and think of the kind of decisions the President of the United States will have to make today.

When you go about making a decision you try to find out as much about the situation as you can. Then you see what choices of action you have. You may in your mind try to imagine the consequences of each possible choice of action. Comparing the consequences you try to arrive at a decision as to what choice suits your purpose best.

In the case of getting up in the morning, you know that you need time for getting ready, eating breakfast and getting to work. You know that if you are late you will need an excuse and have to go to the office. On the other hand, if you were up late the night before you might want some extra sleep. You consider the situation and try to decide.

In simple or unimportant decisions it doesn't pay to spend too much time thinking about it. However, in the kind of decisions that have to be made by scientists, engineers, business men, judges, doctors, political leaders, military men and others, it pays to spend as much time as necessary or as much time as you can.

It may come as no surprise to you to learn that scientific thinking and especially mathematics can be of great assistance in making decisions. However, it has only been since giant electronic computers became available that it became practical to look for a systematic mathematical way to arrive at decisions. In many important decision problems there are hundreds or thousands or even millions of separate things to be decided to reach the final answer. The number of possible choices is too large to look at each possible one individually, even with a computer. The number of choices may even be infinite.

What mathematicians and mathematics do is first, replace the real situation by a "mathematical model". We will find out more about such models later. For now you may think of it as a description of the real situation using mathematics in which details you think are unimportant have been left out. Second, the decision to be made is stated as a special kind of mathematics problem. Third, mathematics then provides special methods for solving the problem in a reasonable amount of time even though there may be an infinite number of possible choices.

Let us look at the kind of a decision problem faced by business men. Suppose you are president of a division of a large corporation called "General Engines". How many cars and how many trucks should be scheduled for the next year's production to make as large a profit as possible? Profit is the money left from the sale of the cars and trucks after all the costs in making them have been paid for.

There is of course a lot of information you will need to have, but as a starter suppose that you know that this year's profit on each truck when sold is $400 and each car when sold gives a profit of $300. You might think that it would be most profitable to build only trucks since each truck brings $100 more profit than each car. But it is not that simple. For one thing each truck uses more steel and if the total supply of steel is limited, you may be able to build many fewer units of trucks than cars and so your total profit may be not as great if you decide to make only trucks.

A little more inquiring turns up the information that each car uses approximately $1\frac{1}{2}$ tons of steel and each truck approximately 3 tons. Also you learn that the total amount of steel available to your division next year will be approximately 975,000 tons. Can you reach a decision? Let's see. If you use all the steel for cars, you can make 975,000/1.5 units or 650,000 cars. At a profit of $300 each, you get a total profit of $195,000,000. On the other hand if you make only trucks, the steel would be enough for 975,000/3 units or 325,000 trucks. At a profit of $400 each, the total profit would be $130,000,000. It looks as if your decision should be to manufacture only cars next year. In fact you wonder why your division of General Engines made any trucks last year at all. However, since the previous president is a pretty smart fellow (in fact he is your boss) you suspect there must be more information that you need to have.

A call to the manager of production asking if he could turn out 650,000 cars next year turns up the fact that the factories have a limited capacity and that at most a half a million units of cars or trucks or both can be turned out in a year.

This changes the situation. Let's see what results if *all* of the units are cars. This gives a total profit of 500,000 × $300 or $150,000,000. But since there is going to be enough steel to make 650,000 cars, a lot of steel, in fact 150,000 × 1.5 or 225,000 tons is going to be unused (remember the factories can only turn out 500,000 units). On the other hand all the 500,000 units cannot be trucks since there is only enough steel on hand for 325,000 trucks.

You, as president, have identified two seemingly possible choices. Use all the steel to make 325,000 trucks at a total profit of $130,000,000 (this does not use the total capacity of the factories to produce). Or use the total capacity of the factories, to produce 500,000 cars at a total profit of $150,000,000 (this does not use all the steel). It would seem that the second choice is the one to follow.

At this point you call up the manager of manufacturing and ask him what the production plan for the *current* year is. You find out that 90,000 trucks and 410,000 cars are being made and the factories are in full production. A little calculation shows the following. The profit on the trucks is 90,000 × $400 or $36,000,000 . The profit on the cars is 410,000 × $300 or $123,000,000 . The total profit this year will be $159,000,000 if all the cars and trucks are sold. This is $9,000,000 more than if you just made cars. Should you go ahead with the decision to make the same number of trucks and cars again next year as this year? After all, the plant is being used to capacity. But how about steel? The amount of steel used would be 90,000 × 3 = 270,000 tons on the trucks and 410,000 × 1.5 = 615,000 tons on the cars. This totals 885,000 tons which is within the limit of 975,000 tons that will be available next year. Of course 90,000 tons of steel will be unused. However, if only cars were made, you recall that 225,000 tons of steel would be unused. Therefore, the present production plan is certainly to be preferred from the point of view of steel used, as well as profit.

Before going ahead, however, you decide to see what the situation was like when the decision was made a year ago for the current year's production. Getting the old figures out of a file you find that last year the amount of available steel for the current year was estimated at 900,000 tons. The present plans call for using all but 15,000 tons. However, next year there will be an extra 75,000 tons available and so the surplus if you stick to this year's production plan will be 90,000 tons. Shouldn't you be able to make use of this extra steel?

At this point you decide to make a list or table showing the choices so far and the profit resulting from each.

Table

CHOICE NUMBERS	NO. OF CARS	NO. OF TRUCKS	UNUSED STEEL (tons)	UNUSED CAPACITY (units)	TOTAL PROFIT (millions)
1	500,000	0	225,000	0	150
2	0	325,000	0	175,000	130
3	410,000	90,000	90,000	0	159

Inspecting the table you see that decreasing the number of cars produced from 500,000 to 410,000 (and producing trucks instead) decreases the amount of unused steel and increases profits. However, if you continue to decrease the number of cars, say to zero, then profits decrease too. This suggests that somewhere in between 410,000 cars and zero cars there may be still room for improvement.

SUGGESTION: HAVE A CONTEST IN YOUR CLASS TO SEE WHO CAN FIND THE BEST PRODUCTION PLAN.

You decide to test this idea by trying a production schedule calling for 400,000 cars and 100,000 trucks. A little arithmetic shows that 900,000 tons of steel will be required with a total profit of $160,000,000. This increases profits by $1,000,000 and would leave only 75,000 tons of steel t year's end. You are delighted at being able to improve on the current year's record. But you still feel troubled about not using 75,000 tons of steel. So you call the Director of Research to get some expert help on the problem and tell him that you need an answer as soon as possible. To your surprise he sends a mathematician to your office.

The mathematician listens while you explain the situation and looks closely at your table and your latest plan which will make a profit of $160,000,000. "You have done very well", he says. "In fact, the most profit you can make is $165,000,000." You are more surprised than you admit and ask him what decision leads to this result. The answer he gives you is to manufacture 350,000 cars and 150,000 trucks. When you check the arithmetic you find that not only will the factories work to capacity, but all of the steel will be used.

Before calling your boss and telling him this decision you ask the mathematician if he is sure that you can't do better. He assures you that you can't and explains as follows. You will be manufacturing 500,000 units, the total allowed and so you cannot make any more units. Therefore if you make one more car you must make one less truck. If you do, you will lose $100 profit and have $1\frac{1}{2}$ tons of steel left over. On the other hand, to make one more truck would require steel you don't have.

This satisfies you for the moment but you want to know how he was able to decide so quickly on the best production schedule. His answer contains a lot of mathematics you never knew before. We will cover this in the rest of the chapter. But perhaps you would first like to know some of the other things he had to say about your decision making problem.

First of all the mathematician explained that this kind of decision problem comes up in many different situations so often that people have given this kind of problem a special name. It is called a linear programming problem. In this one you had two quantities to determine, the number of cars and the number of trucks. In other situations there may be thousands of quantities to determine and a computer would be necessary to solve the problem.

While 350,000 cars and 150,000 trucks furnished the best answer to the problem, the mathematician cautioned that the information being used might still not be complete enough. For example, if "General Engines" would be willing to build a new factory for your division so as to increase the number of units that can be produced it might be possible to increase profits still further. Remember that 650,000 cars would use no more steel than available and would increase the profits to $195,000,000. Of course the cost of the new factory and additional salaries would have to be taken into account and this would reduce the profit on the additional cars built so that it might not be profitable. Also you have not taken into account that materials other than steel might be limited in supply, which could affect your production plans. Many other complications could be taken into account in trying to arrive at a decision. Some might be important, others have only a minor effect.

SUGGESTION: HAVE THE CLASS MAKE A LIST OF OTHER FACTS THAT MIGHT BE TAKEN INTO ACCOUNT IN DECIDING HOW MANY TRUCKS AND CARS TO BUILD. INDICATE BY THE NUMBERS 100 , 10 , 1 HOW IMPORTANT EACH FACT IS THOUGHT TO BE. (100 -- VERY IMPORTANT; 10 -- SOMEWHAT IMPORTANT; 1 -- NOT IMPORTANT)

Notes on Section 2

This is the second section of a chapter on linear equations and inequalities. Here a mathematical model involving equations, inequalities and geometry is provided for the production problem previously presented in verbal and simple arithmetic form. The production problem is reduced to maximizing a linear function subject to linear inequality constraints and the solution is found through examination of the geometric model. The result is that in one specific situation, the student encounters and deals with a variety of mathematical problems related to linear systems. The stage is then set for a return to these problems on their own ground divorced of a specific context to state the problems in more general terms and gain technique in their solution.

The treatment in this section is especially sketchy toward the end. It needs more thought, writing and time.

2. The Mathematical Model.

We are now going to translate our production problem into a mathematical one. First, we must replace the words we have used to describe the situation in ordinary language by mathematical symbols and relations. This will provide a mathematical model. It is only a model of the real situation because (1) we can never list and include all the facts, only those facts that we think are most important, and (2) we cannot know the exact relationships in the real situation, so our mathematical relations will only be approximations to the real life situation.

A great advantage of a mathematical model is that you can do "experiments" with it just with pencil and paper or computer. You can say, "What would happen if such and such were done?" Then, you can carry out the mathematics and find out what the model predicts. You don't have to build something in a laboratory and test it, or wait until it happens in the real world. Often a mathematical model is the only way to get such information when no laboratory experiment is possible. For instance, when you want to determine the route to be travelled to the moon by the first manned spaceships.

If our model is complete enough it will provide a good enough approximation to the real life situation so that we can rely on the answers it gives us. Of course the best test we have is to compare the predictions made by the model with the real situation and see how well they agree. Eventually this must always be done. If the agreement is poor we may have to add more features to the model. You can see that many different models can be made for the same real life situation just as an artist can depict a scene in many different ways.

We first introduce symbols for the quantities we need to find in the production problem. To begin with, let C = number of cars to be produced and T = number of trucks to be produced. The first fact we can express is that the profit made by producing C cars is $300\ C$ dollars. The profit made by selling T trucks is $400\ T$ dollars. The total profit then if we produce C cars and T trucks is $300\ C + 400\ T$. Let us use the symbol P to stand for the total profit in dollars. Then

$$300\ C + 400\ T = P\ . \tag{1}$$

This we will call the profit equation. It is a part of our mathematical model.

It is often useful and particularly in decision problems to think of our model geometrically. If we interpret C and T as rectangular coordinates in a graph and take for P a particular value, say $P = 150{,}000{,}000$, then equation (1) represents a straight line.

QUESTION: WHAT IS THE SLOPE OF THIS LINE?

The model of the profit relation then becomes the straight line segment a a' in Figure 1. The segment is all of the line (1) we are interested in since both C and T cannot be negative; that is, in our real situation we can have the number of cars and trucks positive or zero but not negative. This means that our whole geometrical model must lie in the first quadrant.

Looking at the points on the profit line a a' we see that the point C = 500,000 , T = 0 lies on it. This corresponds to the fact that these values lead to a profit of $150,000,000. As we have previously seen, this is a possible production choice. Our model tells us something new, however: every point (C,T) on a a' satisfies the equation

$$300\ C + 400\ T = 150{,}000{,}000$$

and so gives a profit of 150,000,000 . For example the point (300,000 , 150,000) shown in Figure 1.

QUESTION: Is C = 300,000 , T = 150,000 a possible production plan?

Each time we find or choose a value for the total profit P we will get a new position of the profit line. For example if P = 130,000,000 , we get the segment b b' in Figure 2, corresponding to points on the line

$$300\ C + 400\ T = 130{,}000{,}000\ .$$

Compare b b' and a a' as shown in Figure 2. The geometrical model suggests that they are parallel. This implies that changing the total profit P moves the profit line parallel to itself. If P decreases, the line moves in toward the origin O . If P increases, the line moves out away from the origin O . That the lines obtained by varying P are all parallel is confirmed from the mathematical model expressed by equation (1) . The slope of this line (- 4/3) is fixed by the coefficients 300 and 400 and is not affected by the value of P .

We notice that the point b in Figure 2 corresponds to the production plan where we manufacture only trucks, for C = 0 and T = 325,000 at that point. The interesting fact we learn from our model is that all the other points on b b' also yield the same profit of 130,000,000 . For example C = 300,000 , T = 100,000 .

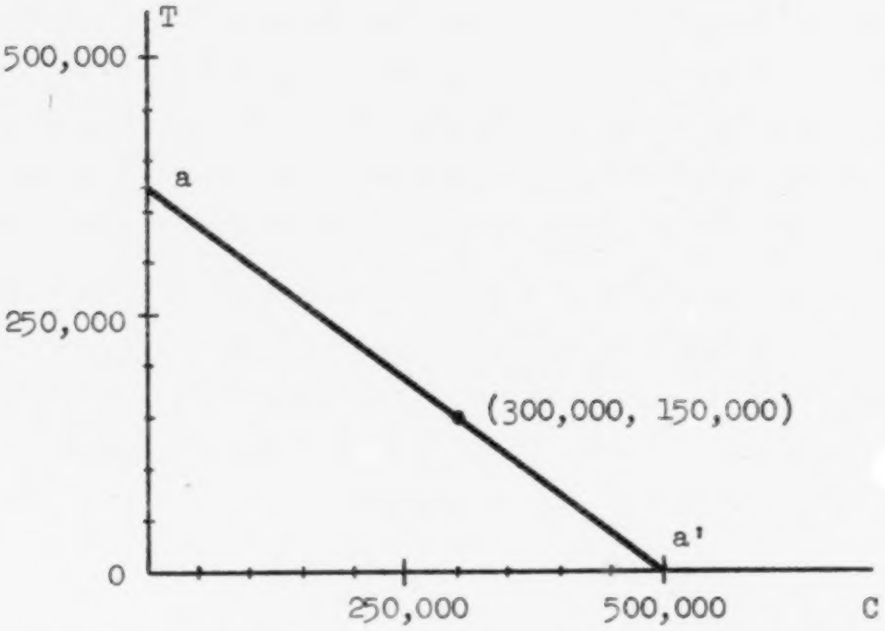


Figure 1. 300C + 400T = 150,000,000

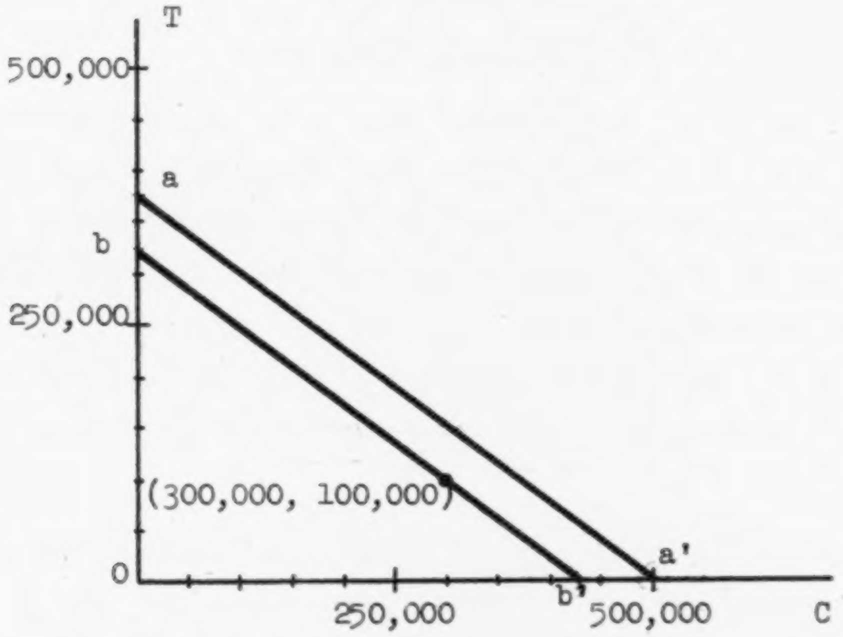


Figure 2. Profit Lines for P = 130,000,000 and 150,000,000

QUESTION: Is $C = 300{,}000$, $T = 100{,}000$ a possible production plan?

QUESTION: Are there any points on $b\,b'$ which do not correspond to possible production plans?

Our geometrical model has suggested an important way of looking at the production decision problem. Each production plan gives a number pair (C , T) . Each such pair corresponds to a point in the first quadrant. Therefore each production plan corresponds to a point (C , T) in the first quadrant. But we know from other conditions in the real problem and production plans such as $C = 500{,}000$, $T = 0$ that not every point in the first quadrant is an allowable production plan. That means our model is incomplete. We must try to put the missing conditions into our model.

Let us see what can be done to incorporate the fact that the amount of steel used cannot exceed 975,000 tons. Since each car requires $1\frac{1}{2}$ tons of steel, if we produce C cars, the amount of steel used is $1.5C$ in tons. Each truck requires 3 tons of steel and if T trucks are produced, the amount of steel used is $3T$ tons. The total amount of steel used is just $1.5C + 3T$ in tons. This amount must be less than or at most equal to 975,000. In symbols we have the relation

$$1.5C + 3T \leq 975{,}000 \, . \qquad (2)$$

This relation we will call the steel restriction; it is an inequality. It states a restriction on the number pair (C , T) . The solution set or solutions of (2) gives us the set of all pairs (C , T) which do not use up more than the allotted amount of steel.

QUESTION: Which of the following pairs are solutions of (2) ?
$(500{,}000 , 0)$, $(0 , 500{,}000)$, $(400{,}000 , 100{,}000)$, $(350{,}000 , 150{,}000)$.

What is the geometrical meaning of (2) ? Or, in other words, what is the graphical model of (2) ? Graphing an inequality is not much different from graphing an equation. For example, $C = 0$ in Figure 1 is the line representing the T-axis. The solutions of $C > 0$ are all points (C , T) in the plane with $C > 0$. These are the points lying to the right of the T-axis. The solutions of $C \geq 0$ are the points lying to the right of or on the T-axis. Similarly, the solutions of $C < 0$ are the points lying to the left of the T-axis.

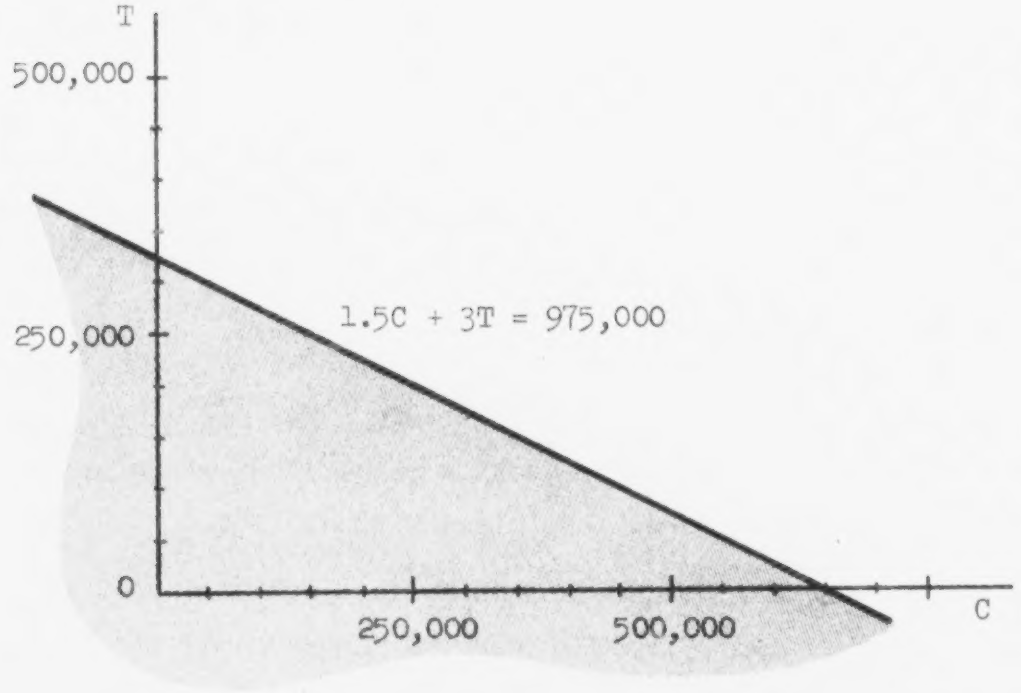


Figure 3. Solutions of $1.5C + 3T \leq 975{,}000$

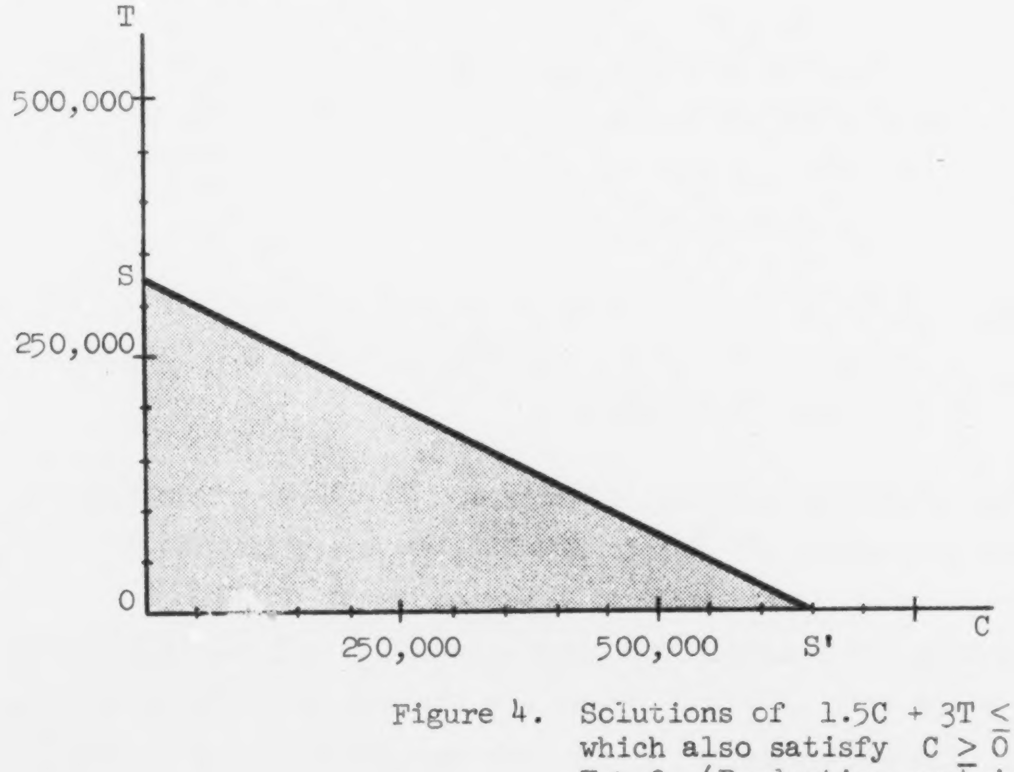


Figure 4. Solutions of $1.5C + 3T \leq 975{,}000$ which also satisfy $C \geq 0$ and $T \geq 0$ (Production restriction due to limit on steel)

To take a few more simple examples, the solutions of $C \leq 2$ are all points (C , T) lying to the left of and on the line $C = 2$. The solutions of $T \leq 2$ are the points (C , T) lying below and on the line $T = 2$.

If the lines are not parallel to the C- or T-axis the situation is the same. The inequality leads us to the set of points lying on one side of the line. We shall study this carefully later in this chapter. For now, however, let us just accept this as being reasonable. In Figure 3 we have drawn the line given by the equation

$$1.5C + 3T = 975{,}000 . \qquad (3)$$

The points (C , T) which are solutions of the inequality (2) all lie on the same side of this line. Which side? We already know some points that are solutions of the inequality, such as $(500{,}000 , 0)$, $(400{,}000 , 100{,}000)$. These lie "below" the line. In Figure 3 we show the graph of the solutions of (2) ; these are the points in the shaded region (which is infinite).

Since we are only interested in positive values of C and T , we have the restrictions $C \geq 0$ and $T \geq 0$ to include in our model. The solutions of $C \geq 0$ are the points to the right of and on the T-axis. The solutions of $T \geq 0$ are the points above and on the C-axis. The points common to these two sets (their intersection) are the points of the first quadrant. Combining this restriction with the restriction imposed by the inequality (2) we must drop all shaded points in Figure 3 except those in the first quadrant. This gives us the set of points contained in the triangle OSS' in Figure 4 including its boundary.

To summarize what we have just found: the set of pairs (C , T) satisfying the inequalities

$$1.5C + 3\,T \leq 975{,}000$$
$$C \geq 0$$
$$T \geq 0$$

consists of the triangle OSS' and its boundary (shaded in Figure 4). This is a mathematical model of the following statement:

> The amount of steel used in the production of C cars and T trucks cannot exceed 975,000 tons.

However, our model is not yet complete. We have not taken into account

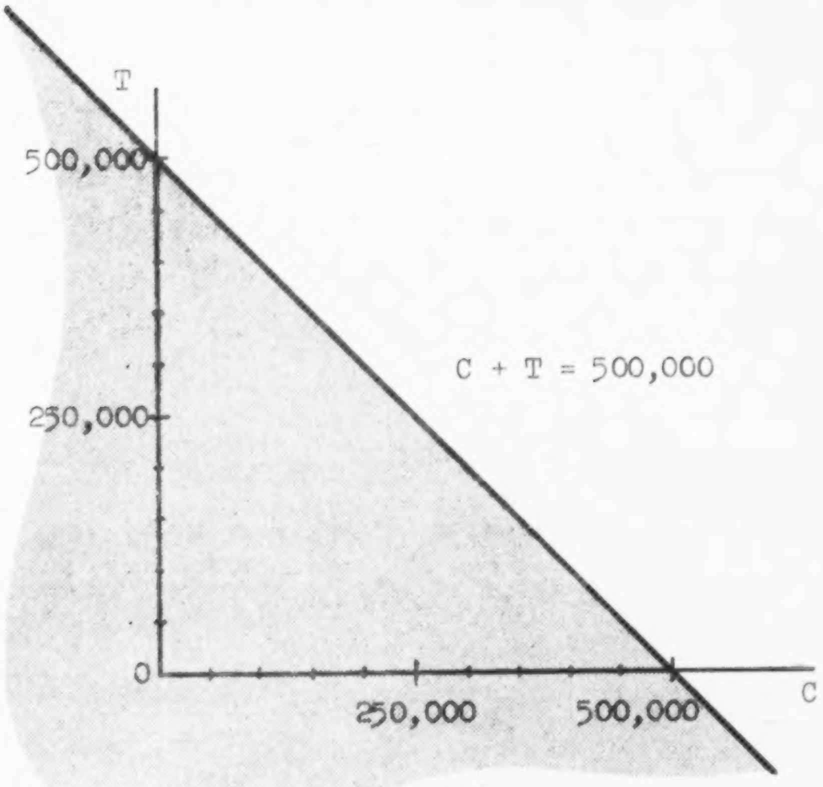


Figure 5.
Solutions of $C + T \leq 500,000$

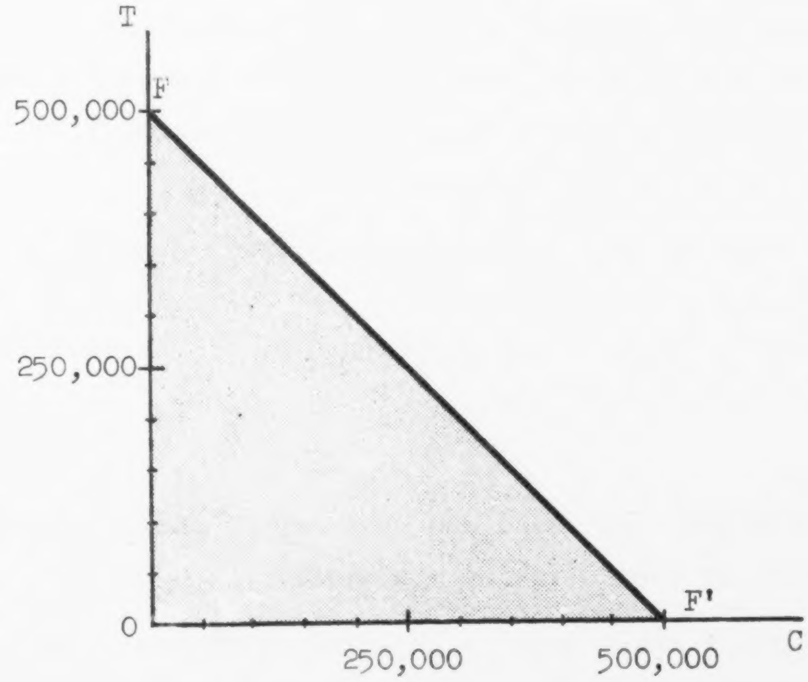


Figure 6.
Production Restriction Due to Factory Limitation.
($C + T \leq 500,000$, $C \geq 0$, $T \geq 0$)

the restriction on plant capacity. For example there are points in OSS' for which the total number of vehicles would exceed 500,000 , the capacity of the factories.

QUESTION: Which of the following points in OSS' correspond to production plans which would exceed plant capacity?

(250,000 , 150,000) , (500,000 , 0) , (525,000 , 10,000)

The restriction on plant capacity states that the total number of cars and trucks cannot exceed 500,000 . The mathematical model of this statement cast as an inequality is given by

$$C + T \leq 500,000 . \tag{4}$$

To find the geometrical model we first graph the line

$$C + T = 500,000 .$$

(see Figure 5)

The inequality (4) admits as solutions all points to one side of this line. These are the points shown shaded in Figure 5.

ASSIGNMENT. Locate the points (250,000 , 200,000) , (100,000 , 500,000) , (400,000 , 100,000) , (600,000 , 300,000) , (200,000 , 300,000) , (400,000 , 50,000) . Which of these points are solutions of (4) ?

If to (4) we add the restrictions $C \geq 0$ and $T \geq 0$, the solution set is reduced to the shaded triangle OFF' in Figure 6.

We can now interpret the totality of possible production plans in terms of our model. The allowable plans for the production of cars and trucks are given by the number pairs (C , T) satisfying the following restrictions:

$$C \geq 0 \; , \; T \geq 0$$

$$1.5C + 3T \leq 975,000$$

$$C + T \leq 500,000 .$$

Geometrically this solution set is the intersection of the solution sets shown shaded in Figures 4 and 6; that is, the set of points satisfying $C \geq 0$, $T \geq 0$ which satisfy both of the other inequalities. The intersection of these sets is shown cross-hatched in Figure 7. It consists of the points contained

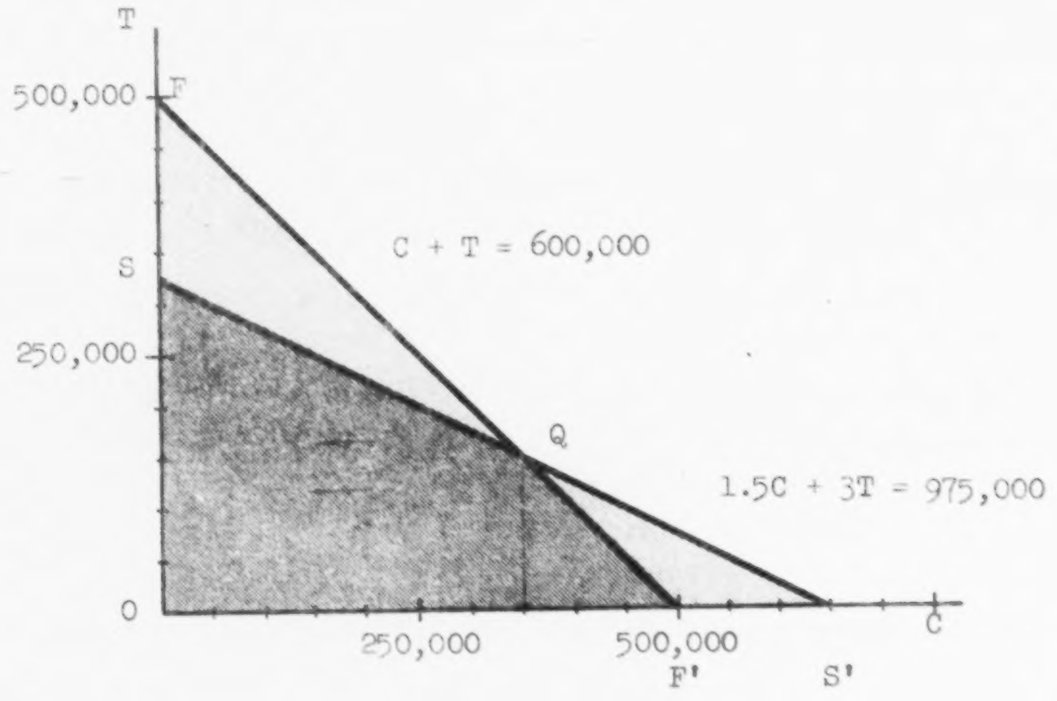


Figure 7.
Intersection of Solution Sets of $1.5C + 3T \leq 975,000$, $C \geq 0$, $T \geq 0$
And $C + T \leq 500,000$, $C \geq 0$, $T \geq 0$ Representing all Permissible Production Plans

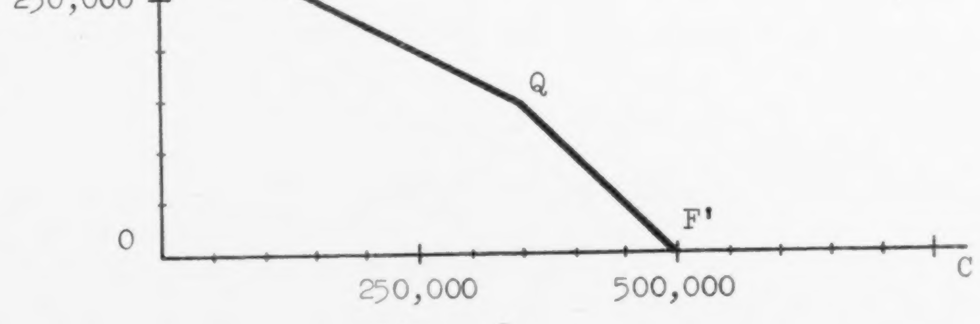


Figure 8.
The Set of all Permissible Production Plans (C, T)

in the four sided figure (quadrilateral) OSQF' and its boundary.

The point Q which locates the intersection of the line segments FF' (the factory restriction boundary) and SS' (the steel restriction boundary) is of particular importance as we shall see. What are the coordinates of this point? The number pair (C , T) at Q can be found from the equations of the lines,

$$1.5C + 3T = 975{,}000$$
$$C + T = 500{,}000$$

on which Q lies by simple algebra. We shall explain these useful techniques later in this chapter. For now, however, we observe from Figure 7 that the intersection appears to be at the point (350,000 , 150,000) . We can check that this is correct by replacing $C = 350{,}000$, $T = 150{,}000$ in the above equations and seeing that the statements are true.

Our mathematical model has led us to the region OSQF' in Figure 8. This set of points corresponds to all production plans which satisfy the restriction on plant capacity and the restriction on steel. But which production plan is best? Which plan gives the most profit? Can we ask this question in mathematical terms? What is the mathematics problem which expresses the production problem and whose solution is the solution of the production problem?

To state the problem, we return to equation (1), which tells us what the profit P is for any choice of C and T . All the choices of production plans are represented by the number pairs or points (C , T) in the set shown in Figure 8. Which of these points gives the largest value to the profit P ? That is, what number pair (C , T) in or on the quadrilateral OSQF' makes P a maximum (as large as possible)?

This is very nearly the precise mathematical statement of the problem. In wholly mathematical language, we want the number pair (C , T) satisfying the conditions

$$C \geq 0 \; , \; T \geq 0$$
$$1.5C + 3T \leq 975{,}000$$
$$C + T \leq 500{,}000$$

for which the quantity

$$P = 300C + 400T \qquad (1)$$

is a maximum.

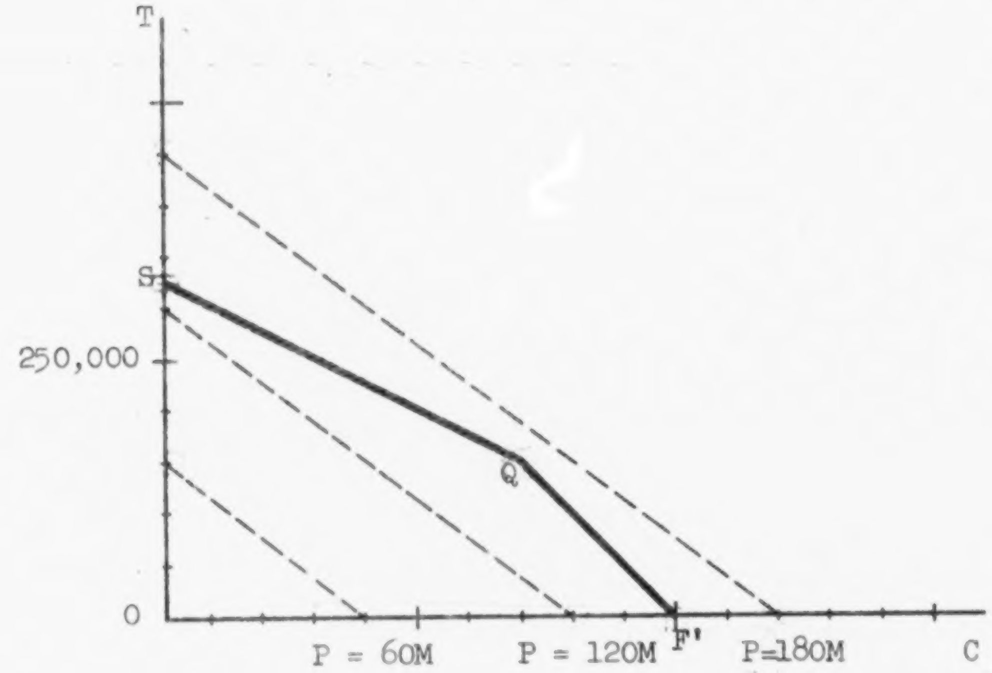


Figure 9.
Profit Lines Intersecting Solution Set for Three P Values

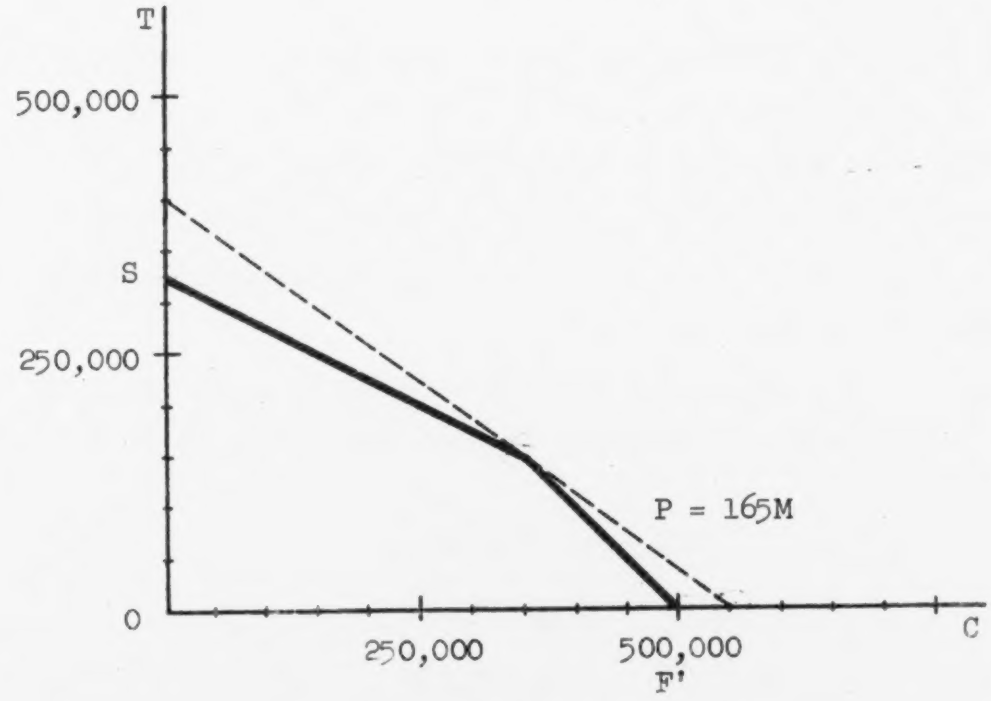


Figure 10.
Intersection of Maximum Profit Line with Solution Set

What is the solution to this new mathematical problem? You will recall that the mathematician in the previous section chose the point Q with coordinates (350,000 , 150,000) as the point for which P is a maximum. How does our mathematical model lead to this solution?

To answer this let us study Figure 9. Here we have shown the solution set and also by dashed lines the lines given by (1) corresponding to three different profits: 60,000,000 , 120,000,000 , and 180,000,000 ("million" is abbreviated by the letter M in the figure). As we saw at the beginning of this section the lines given by (1) with different values of P are all parallel to each other.

The line for $P = 60M$ intersects the solution set in the line segment shown closest to the origin in Figure 9. All points along that segment lie in the quadrilateral OSQR' and so represent possible production plans which lead to the profit of $60,000,000 . If we ask for a larger profit, say $120,000,000 , we plot the line given by the equation

$$300C + 400T = 120,000,000 .$$

This gives the dashed line segment midway between the others. All points along this segment also lie in the solution set of permissible production plans. Therefore, all the points on this segment are permissible production plans yielding a total profit of $120,000,000 .

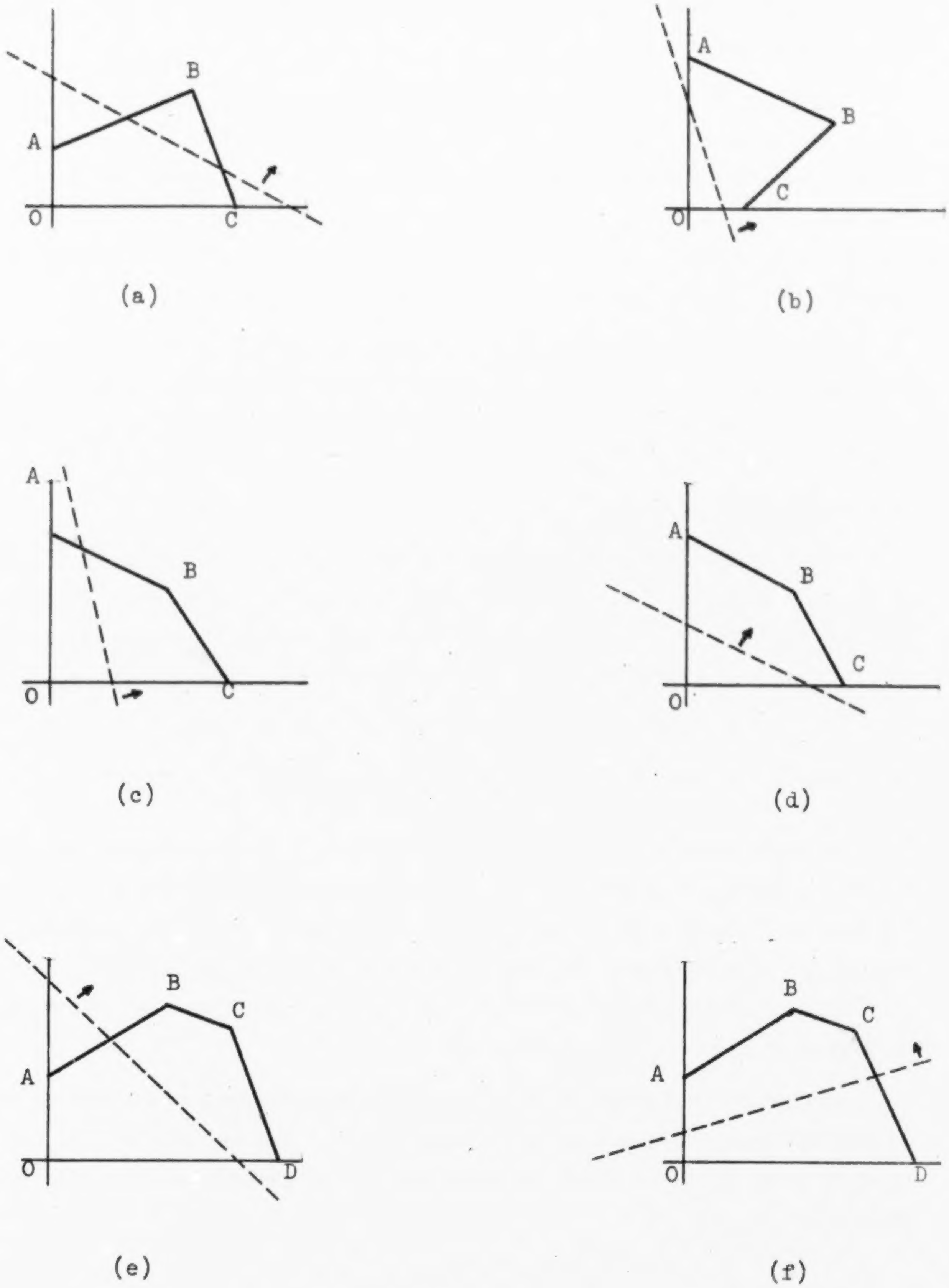
However, if we ask for a total profit of $180,000,000 our model tells us this is not possible. Plotting the line

$$300C + 400T = 180,000,000$$

we get the top dashed line in Figure 9. This does not intersect the quadrilateral OSQF' anywhere. Therefore, there is no production plan which will give this large a profit.

The model suggests the following. The profit line should be moved "as far out as possible", since moving it out corresponds to increasing profit. It can be moved no further when it will just be tangent to the solution set. This situation is shown in Figure 10. Here the profit line just touches the solution set at Q . Since at Q , $C - 350,000$ and $T = 150,000$, the value of P from (1) is \$165,000,000 . Any further increase in P means that the line (1) moves further out from the origin (remaining parallel to the line shown) and so cannot intersect the solution set.

The method of solution corresponds to moving the profit line as far as possible until it is just tangent to the solution set corresponding to the conditions of the problem. The point of tangency gives the solution -- that is, the values of C and T . It can happen in problems of this kind that instead of a single point of tangency, the line will coincide with a segment of the boundary of the solution set. In this case more than one possible solution to the problem exists. However, if the solution set is a (convex)


A
B
C
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(a)
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(b)
A
B
C
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(c)
A
B
C
O
(d)
A
B
C
D
O
(e)
A
B
C
D
O
(f)

Figure 11.
Some Sample Situations. At What Point Does the Solution Occur?

plane figure bounded by straight edges, the maximum can always be found at one of the vertices.

This means that in seeking a production plan we need only check the profit at each vertex of the graph of the solution set. Since there are only a finite (and in our case small) number of vertices this is a rapid method of finding the best plan. In the study of these problems (linear programming problems is the name given by mathematicians) there is a method of solution called the "simplex method" in which the vertices only are used.

In Figure 11 are shown some examples illustrating the mathematical model in geometrical terms corresponding to several decision problems. The closed figure represents the set of possible solutions. The dashed line corresponds to one position of the profit line (or whatever quantity it is desired to maximize).

Increasing profit corresponds to moving the profit line in the direction shown by the arrow. Tell in each case what point (or points) of the solution set maximize the profit. In cases (e) and (f) there are additional restrictions which account for the additional sides in the plane figure.

Notes on Section 3

In this section are cited some of the most common situations reducible to linear programming problems. It is intended to broaden the perspective of student and teacher with regard to the applicability of the mathematics contained in this chapter. Having concentrated on the car-truck manufacturing problem, it is important to break the tie, to emphasize that the power of mathematics lies in its generality.

Following this section will come the mathematical treatment of simple linear systems of equations and inequalities. The end of the chapter can then draw on some of the situations sketched in the present section for problem material.

3. Related Problems.

In the preceding sections we have spent a good deal of time analyzing one particular problem: how much to produce of two products with limited resources so as to maximize the profit. We have said that this is one kind of linear programming problem. Such problems arise in many situations which can seemingly be very different. It is the great power of mathematics that once these seemingly different problems have been analyzed by means of a mathematical model, they are all found to reduce to the same or very similar mathematics problems. Once we learn how to solve these kinds of mathematics problems we are equipped to handle many situations which at first seem completely different.

In the following sections of this chapter we will learn how to solve some of the kinds of mathematics problems which are common to all the different situations. Before doing this, however, we will now describe some of the different situations which lead to linear programming problems and therefore to similar mathematical questions.

a. The Diet Problem.

A diet for losing or controlling weight is to be planned using a number of different foods. Each food contains certain amounts of different nutriments (such as proteins, vitamin C, calcium, etc.) per ounce. Each food also contains a certain number of calories per ounce. The diet requires certain minimum amounts of each nutrient per day. The problem is to find the amount of each food to be included in the diet which will give at least the minimum amount of nutrition but will make the total number of calories as small as possible.

b. Transportation Problems.

A company maintains a warehouse in each of a number of cities. Each warehouse holds a certain number of units of a given commodity (such as refrigerators). Orders come in from dealers in surrounding places. We are given the number of units required by each dealer, the distances from the dealers to the warehouses and the cost of shipping from each warehouse to each dealer. The problem is to decide how to fill the orders: how many units to ship to each dealer from each warehouse in order that the cost of shipping to meet all the orders be a minimum (as small as possible).

A variation of this problem (in wording alone) occurs in military operations planning. A nation maintains a number of naval bases. Each base is the home of a certain number of aircraft carriers, destroyers, etc. At a certain time it is necessary to assemble a task force at each of several locations for maneuvers. The number of ships of each type to rendezvous at these spots is assigned. The distance from each base to each rendezvous location is given. The problem is to decide on the orders for the ships: which destination for the ships from each base in order to have the total travel time for all the ships a minimum. Notice that in this example and the warehouse problem, minimizing time is the same as minimizing distance. Since fuel consumption (for trucks or ships) is proportional to distance travelled, this is also the same as minimizing cost.

An interesting variation is the following: if the task forces have to be assembled as fast as possible (say if there is an international emergency), then the problem is no longer the same. We would not be interested in adding the travel times of all the ships, but in the longest time taken by any ship to reach its destination. This leads to a different type of mathematical problem.

c. Blending Problems.

An oil company through its refining of crude oil produces a certain number of barrels daily of each of several different chemical components of the oil. These components can be blended to make different marketable products such as grades of automotive gasolines and aviation gasolines. These products sell for different prices. We are given the number of barrels of each component produced daily,. the blending rules and the sale prices of the final products. The problem is how much of each product to produce daily to yield the maximum income.

A variation of this problem involves mixtures. As an example, suppose that two mixtures of nuts are to be offered for sale: a regular mix and a party mix. The proportions of the different kinds of nuts used for each mix are prescribed. Also given are the costs per pound of each kind of nut, the total supply of each kind of nut available and the selling price per pound of the two mixes. The problem is to decide how many pounds of each mix to produce out of the given supply so as to maximize the profit.

d. Network Problems.

These are problems involving a network of some kind such as the telephone lines interconnecting cities, roads and highway systems, connections in an electronic circuit and so forth. As an example suppose that special communications cables (say for TV) have to be laid to join a number of distant cities. It is not necessary to lay a cable directly between each two cities as long as some route can be found between them. For example a cable need not join Chicago and Los Angeles directly if there is one from Chicago to San Francisco and one from San Francisco to Los Angeles since these can be joined at a switching station in San Francisco. Given the distances between each pair of cities the problem is to determine which cities to join by cable in order that any city in the network can communicate with any other city and so that the total amount of cable to be laid is a minimum. This is called the shortest connecting network and the same problem frequently arises in the telephone business. If there are n points in the network to be connected there are n^{n-2} possible networks. This number increases very rapidly as n increases and it becomes impossible simply to measure all possible networks. As a linear programming problem the solution is easily found.

In Figure 1 is shown the location of a number of points to be joined by the shortest connecting network and in Figure 2 is shown the solution.

A related problem concerns the maximum flow in a network. Suppose the various cities in a network are joined by telephone "trunk lines", and that each trunk line can handle a certain number of calls. If a trunk is fully used, alternate routes can be found to place a call, using trunks to other intermediate cities. Given the locations of all the trunks and the maximum number of calls each can handle, what is the maximum number of calls which can be made at one time from one city to another, say from New York to Los Angeles?

e. The Assignment Problem.

Suppose there are a number of jobs to be filled and a certain number of people available to carry out these assignments. Each person could be assigned to any one of the jobs, but he is better at some jobs than others.

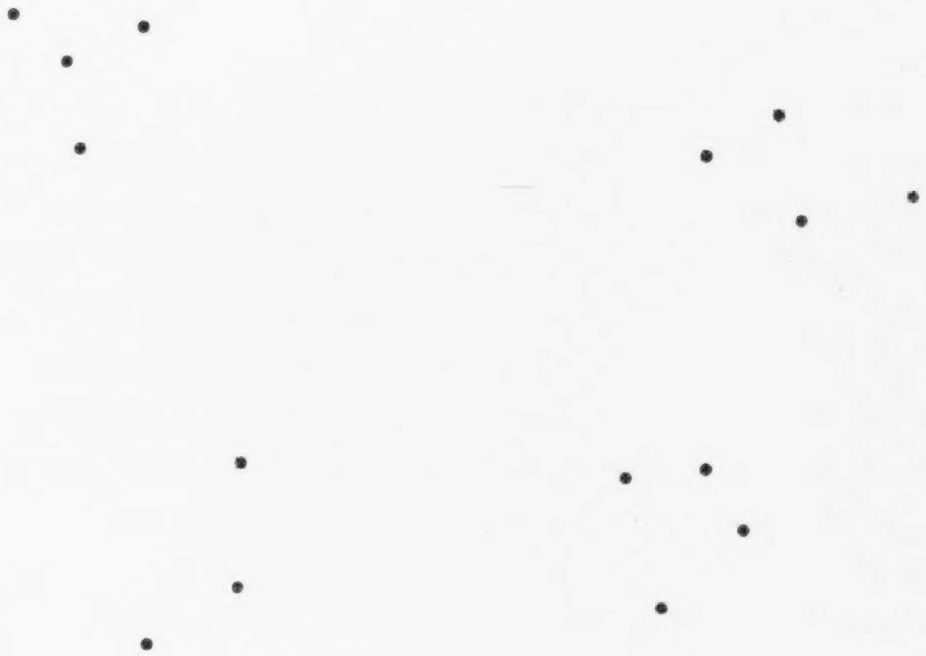


Figure 1.
15 Points to be Joined by a Network

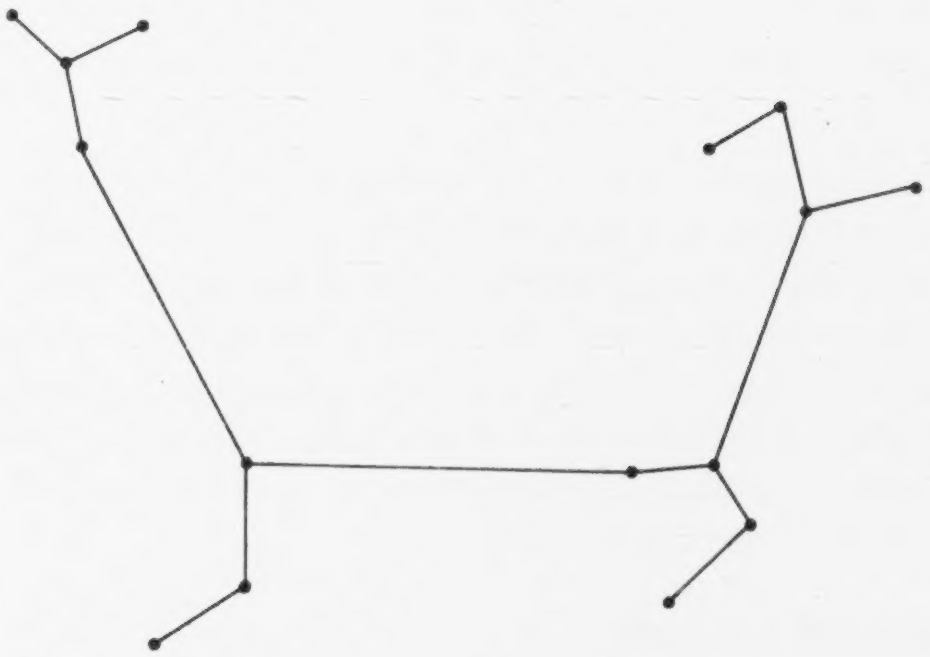


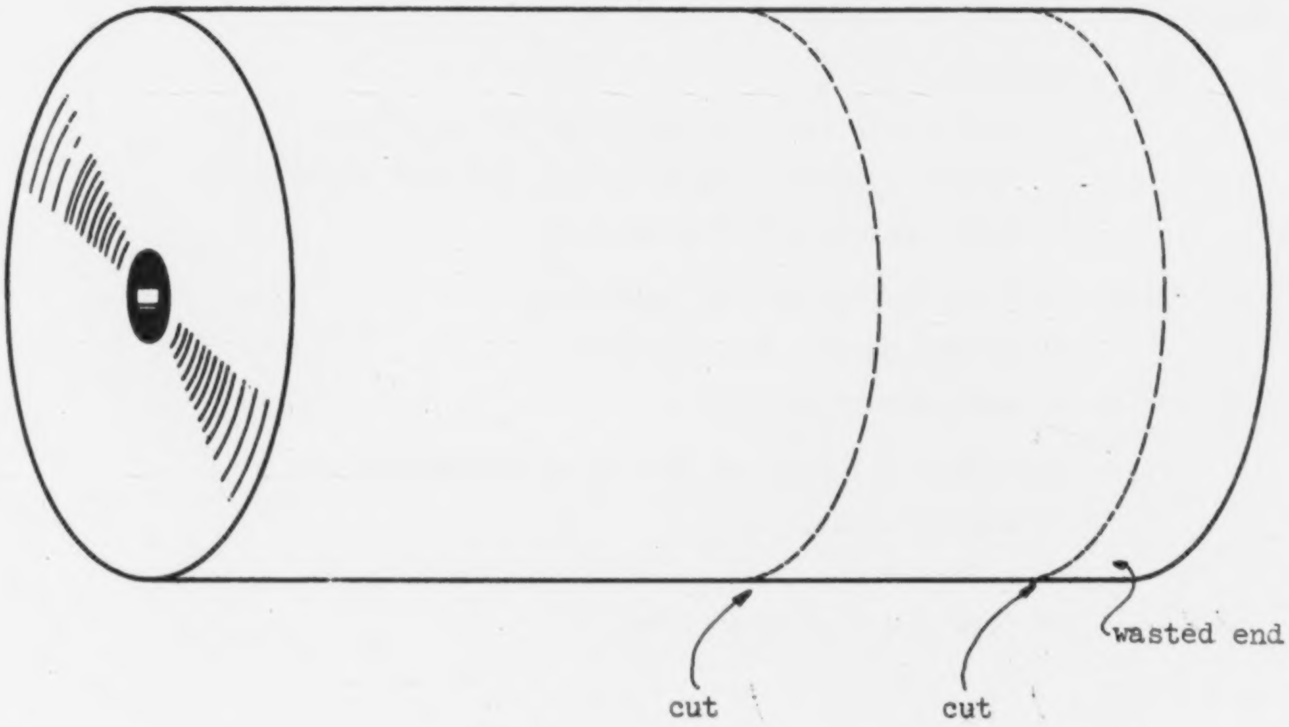
Figure 2.
Shortest Connections Network

Suppose we are given a rating for each person for each of the jobs (say 10 if he is very good at it, down to 1 if he is very poor). The problem is to assign the people to the jobs so that the sum of all the ratings of the people in the jobs they are assigned is a maximum.

f. The Trim Problem.

Paper mills produce paper in large rolls in certain standard widths only. Customer orders come in specifying any intermediate widths they desire and the number of rolls of each. These widths are obtained by cutting or trimming the rolls. Figure 3 shows the location of cuts along a roll which might occur.

A single roll might be cut in many ways to fill different orders. The problem is, given the widths of the standard rolls and given the customers' orders, how should the rolls be cut so that the amount of paper wasted (unused ends of rolls) is a minimum?



4. Outline for Mathematics Sections of Chapter on Systems of Linear Equations and Inequalities.

The preceding three sections of this Appendix are Sections 1, 2 and 3 of a chapter on systems of linear equations and inequalities. The object of the succeeding sections of the chapter is to develop the mathematical ideas and skills which underlie the formulation and solution of the particular problem already considered in detail. No attempt has been made to do this in detail since there is no need for a new approach or for much, if any, dovetailing with the previous material. The presentation can be quite straightforward. The motivation and the experience are already present. The need now is first to cover the mathematics with dispatch and at a level of explanation in keeping with the students intuition and experience; second, to provide drill and third, to provide good problem material at the end, roughly equivalent in difficulty to the original problem which began it all (see Section 3 of this Appendix for source material).

Section 4. Linear Equations.

4-1. Graphing a line.
- -- stress solution set description of points (x,y),
- -- relate slope to coefficients, perhaps informally.
- -- when are two lines parallel?

4-2. Graphing System of Two Equations.
- -- solution set possibilities.
- -- approximate solution.

4-3. Algebraic Solution of System of Two Equations.
- -- substitution.
- -- elimination.
- -- flow chart of algorithm.

Section 5. Linear Inequalities.

5-1. Graphing an Inequality.

-- $x > a$, $x < a$, $x \geq a$, $x \leq a$.
$y > b$, $y < b$, $y \geq b$, $y \leq b$ (a,b numbers).
-- stress solution set description.
-- $x + y = 1$, $x + y > 1$, $x + y < 1$, etc.
-- linear inequality as half-plane.

5-2. Graphing System of Two or More Inequalities.

-- intersection of half planes $x \geq a$, $y \geq b$ (boundaries parallel to axes).
-- intersection of two half planes of any orientation (emphasize geometry, not algebra).
-- intersection of three half planes to define triangle (emphasize geometry, not algebra).
-- distinguish unbounded and bounded situations.

Section 6. Optimization.

6-1. Linear Functions Defined on Plane Domain.

-- $f: (x,y) \rightarrow ax + by$.
-- intersection of solution set for $ax + by = P$ (a, b, P given numbers) with solution set corresponding to system of inequalities.
-- effect of varying P on intersection.
-- maximum or minimum of P at extreme points of set (informal).

Exercises: A set of exercises at close of chapter to explore applications such as considered at the beginning of the chapter. Types of problems are included in Section 3. Consult many existing sources for actual numerical problems (The Mathematics Teacher articles; Kemeny, Snell, Thompson; Richardson; Dorn and Greenberg; etc.). Emphasis on finding solution sets corresponding to various linear restrictions and combinations of restrictions; secondarily on optimization (to be returned to in a higher grade).

5. Suggestions For the Development of the Topic "Systems of Linear Equations and Inequalities "in Grades 7, 8, 9 and 10.

The written material in the first 4 sections of this Appendix (car-truck production model through accompanying mathematics) was intended for inclusion in the 7th grade syllabus. This was based on the suggested outline (Summer 1966). It goes far beyond what was included in the 7th grade material written during the year. However, new proposals indicate changes in the 7th grade syllabus making it likely that the material will be more appropriate for the 8th grade. If this is the case, it is recommended that the material outlined on pp. 316, 317, 318 (July 1966 outline) together with the material in the first 4 sections of this Appendix be used as a basis for the 8th grade chapter.

This would leave the 7th grade still not settled in this topical area. It would seem reasonable to cover single equations and inequalities emphasizing solution sets, graphical understanding and problem situations. The 8th grade material might be anticipated without going into details of technique.

There are several objections to the present Chapter 7, in Grade 7. First, the problems are almost classic examples of the kind we want to avoid. They are unreal and uninteresting. They also come last instead of first. Second, the method of solution of equations based on "boxes" does not seem a good one to introduce although it is clever. It will not be followed up in later work. It eats up a lot of space and time. It also is apt to be confused with flow charting since it involves boxes and arrows. Finally, it does not seem as if enough ground is covered. One could, for example, consider the maximum and minimum of a linear function on an interval, that is, $f: x \to ax + b \mid c \leq x \leq d$ to find Max f and Min f . These occur at the ends of the interval (extreme points) which leads well into the optimization problems to be encountered in 8th and 9th grades. Also some work with systems ought to occur.

Assuming that the 8th grade is based on the kind of material in this Appendix and in the outline on pp. 316-318, we can proceed to a 9th grade syllabus in this topic. It is suggested that a natural topic is convexity and particularly polyhedral convex sets as defined by the intersections of half-planes. The problem of maximizing or minimizing linear functions defined on such sets should be the focal point of the treatment and it should be proven that these occur at extreme points of the set.

This topic would be a good bridge to the vectors and linear algebra of the 10th grade and is of considerable interest both mathematically and for applications.

Linear programming examples ought to form the bulk of the problem material involving applications. However, one should not go beyond two variables and so there is no need to present the simplex method. However, this would be a good topic for the 10th grade together with more variables and a general algorithm for solving systems of linear equations (Gaussian elimination). Flow charting should be used in the 10th grade to specify the algorithms and here is an excellent point to emphasize numerical analysis and introduce computer programming (see Dorn-Greenberg, Chapters 1 and 2).

6. Some Thoughts on the "Student's Manual" to Accompany the Teacher's Text for Model-Motivated Mathematics (3M).

There seems to be some basis for the claim that students in Grades 7-12 (and beyond) do not read, or read only the minimum required to do their assignments. It also seems that a 3M presentation is largely in the hands of the teacher and it is for her or him that the text will be written. Accordingly, what is it that is to be produced in print for the student?

As a start, here is a list of some possible purposes of the material printed for the student, assuming that what goes on in the classroom thoroughly involves the student as an active participant:

1. To consolidate the gains in the classroom and summarize content for clarification and review.
2. Provide additional practice with more and varied problem material.
3. Provide special material for more talented students.

One can (probably perilously) jump to the conclusion that no expository text material of the usual kind is needed. This does not mean that a list of problems of the usual kind is what is needed either.

An acceptable and perhaps even ideal solution for our purposes is suggested by the Learning Mathematics books. (Shropshire Mathematics Experiment - Penguin Books; see SMSG library). Here we apparently find no exposition, only exercises.

However, on closer inspection we find that the exercises in each section contain all the instructional material. These are so arranged and graded that in working through them the effect is cumulative instruction combined with problem solving. Because the exercises constantly involve new ideas and techniques they are challenging and can be expected to develop the student's ability to reason mathematically.

One of the impressive features of these books is their casual (one might say, cool) and concise way of introducing new material. This is almost forced by the format. As a result there is genuine and visible progress from exercise to exercise, page to page.

Clearly, to devise an exercise book of this kind is a most difficult thing to do. The reason is that we usually write a chapter of a text first and then supply the exercises. At this point and only at this point, however, do we know what the chapter contains. If we want to produce student material for a 3M type of course perhaps we ought then to throw away the text (or maybe give it to the teachers) and redo the exercises to incorporate what is essential in the text. This would be doing for the student in a conscious way what he instinctively tries to do for himself.

It might be worthwhile to produce a sample of this kind of student material, perhaps to accompany the 3M kind of text material (mainly for teachers) that is in this Appendix on linear systems. This kind of material in the students' hands for homework would seem to be an ideal complement to the classroom situation in a 3M style course. A test of student use, acceptance, and learning from such material would be interesting.

7. Two Pairs of Physically Stated Dual Linear Programming Problems.

Two items worth doing or considering, among the others suggested in Grade 9, Chapter 3, Systems of Sentences and Optimization, is (1) to pick some problems which are really isomorphisms of each other, e.g., the single commodity transportation problem (steel), the naval task force problem (Section 3 of this Appendix), and the personnel assignment problem and (2) to pick some problem pairs which are dual problems. While it is not recommended that algebra of duality be gone into, a remark about duality being a notion appearing frequently in various places in more advanced mathematics is probably in order. The fact

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Table 1. Number of Children in Each Book of Programmed Mathematics Curriculum, Shown by Weeks

Book	April 3-7	April 10-14	April 17-21	April 24-28	May 1-5	May 8-12	May 15-19	May 22-26	May 29-June 2	June 5-9	June 9-1
6C	3	1	1								
7A	4	1									
7B	2	2	2	1	1						
8	1	2	2	2*		1	1	1	1		
9A	2	3	2	3	1						
10	1	1	3	5		1				1	1
9B		1		1	4	1					
14A		1	2	2	4	1	1				
14B	1				1	6	3				
15A	6	1			1						
15B	4		1			1	1				
16	14	7	1		2		5	5	2	1	
17A	6	6	8	7	1	3	1	5	4	4	4
17B	6	20	8	5	2		2	1	3	3	3
18A		4	8	11	6**	2	2	2	3	2	1
18C			12	7	5	5	.	5	1	1	2
18B				8	14	6	7	3	5	2	2
19A					4	8	4	4	3	3	3
19B					5	8	4		2	3	1
20A					1	4	6	13	7	3	4
20B					1	5	9	6	5	7	8
21A						1	2	1	1	1	1
21B							5	5	5	1	1
22A								2	6	6	
22B									4	4	9
22C										2	1
23A										1	2
23B										4	1
24A										1	4
24B										2	2
25A											
25B											2

* Two children enrolled, started in Book 8.
** One child enrolled, started in Book 18A.

In a well planned overall operation the inequalities are actually equalities. The total production of the furnaces is also the total of input capacities of the foundries. We will assume these conditions. Thus

$$\sum_j x_{ij} = a_i \,, \tag{6}$$

$$\sum_i x_{ij} = b_j \tag{7}$$

and

$$\sum_i a_i = \sum_i b_j \,.$$

Dual.

Entrepreneur enters the scene again. Because of a certain tightness in the market a grey market in steel has arisen. The entrepreneur, however, knows of certain alternative sources of supply at the foundry sites and of certain customers demanding steel at the cities where the furnaces are located. Thus he offers to take the steel off General Steel's hands at the furnace sites at a price of p_i per ton and supply "grey market" steel at the foundry sites j at a price of q_j per ton. Clearly no deal results unless

$$-p_i + q_j \le c_{ij} \;; \tag{8}$$

where p_i and q_i are non-negative, obviously. The entrepreneur must supply b_j tons at foundry j and buy a_i tons at furnace site i. Thus his enterprising venture with General Steel brings him a sales return of

$$\sum_i (-a_i)p_i + \sum b_j q_j \tag{9}$$

which he wishes to maximize.

We note once again that the coefficients in (5) become the right hand sides of the inequalities (8) and, writing (6) as

$$-\sum_j x_{ij} = -a_i \tag{6'}$$

we see that the right hand sides of (6') and (7) become the coefficients of the objective function (9). Also the matrix associated with the system (6') - (7) has as its transpose the matrix associated with (8). Thus again we get a physically stated dual problem.

The model making aspects of these problems should be made explicit. The assumption of negligence of certain characteristics to make the mathematical solutions feasible should not escape mention. For example in both pairs of problems the commonly occurring situation of lower unit prices for large quantities (either bought or shipped) is assumed not to happen. (Actually with more sophistication this can be dealt with also.) Another assumption in the steel shipment problem was that only one grade of steel was shipped. Usually there are different kinds, making a multi-commodity problem or, instead, possibly several separate problems.